An Enriched Model of Strategic Voting under Uncertainty

Henri Surugue[1] and Sébastien Destercke[2]

[1] *Sorbonne Université - Nukkai, LIP6*
[2] *UTC Compiègne, Heudiasyc*

Abstract

We present a new strategic voting model where we use uncertainty representation to model preferences. Specifically, we use probability sets as uncertainty representations, together with lower and upper expected utility gains to take strategic decisions. Focusing on belief functions in particular, we demonstrate that this very expressive model includes in one sweep many existing models based on probabilities, sets or incomplete preferences. Additionally, we generalize several well-known convergence results from the literature to this broader representational setting. Furthermore, we illustrate how this model can capture more realistic scenarios for practical applications but also raises theoretical challenges.

Keywords: Computational social choice, Voting theory, Strategic voting, Uncertainty, Beliefs functions.

1 Introduction

Strategic voting [21] arises when voters have an incentive to misreport their true preferences to obtain a more desirable result. The Gibbard-Satterthwaite theorem [10, 30] shows that no reasonable voting rule is immune to strategic manipulation. This makes it desirable to formalise the notion of strategic votes and investigate their consequences, first in a single-step, single-voter setting and then in multi-steps, iterative settings. This can be done in various ways: formulating the problem as a strategic game [23] or as a Bayesian decision problem [27, 13]. These approaches raise the question of how voters act strategically as well as the role of polls, and show that understanding how information impacts voter opinions is crucial [9, 29].

Several reasons justify the need for an uncertainty-based approach to polls, stemming either from the polling process itself or from the voters themselves. Indeed, building polls, even when done honestly, comes with uncertainty because you cannot ask everyone's preferences (see the case of dishonest polls [26]). We can also interpret uncertainty from voters as they may be unsure about their final preferences or votes, especially if the elections are still far in the future. However, whatever the source of uncertainty, it ultimately translates into uncertainty about the scores under the Plurality rule, as these are the only relevant pieces of information for strategic behavior under this rule. Models of uncertain polls either consider a distribution over possible score vectors [27], or uncertain neighborhood of scores [20]. Uncertain voters, on their side, are commonly modeled by the means of incomplete preferences [3, 8, 32]. Within such models, strategic vote changes have been studied either as a single-step chance for a voter [27, 3] or as a multi-step iterative procedure [20, 32, 18].

Our contributions in this paper are the following:

- We consider modeling voting uncertainties by sets of probabilities, a very expressive uncertainty model that has several advantages: it includes in one sweep set-based and probabilistic approaches, unifying them under a common umbrella; it allows voters to better express their uncertainty; it can benefit from strong decision theoretical foundations. Section 2 introduces the representation, and provides several examples showing how its expressive power can help in poll or voter uncertainty modeling;
- In Section 3, we introduce our decision-theoretic part, providing several decision rules that include as specific cases some of the rules we mentioned [27, 3], allowing to reinterpret them within our framework and to complement them with higher expressive power. Although we will not perform empirical studies in this paper, such a higher expressiveness could also be useful in descriptive studies, as more voter behaviours can be described by the model;
- Finally, our study culminates with Section 4 that considers iterative, multi-steps frameworks, for which we show that convergence results similar to the one of Meir [22] can be obtained with very natural, simple models of uncertainty that also present computational advantages, namely possibility and inner measures.

Note that this paper focuses on plurality voting, where the information is summarized by candidates' vote scores.

2 Uncertainty model on votes

We start by introducing the elements of our newly proposed model. As the model itself is one of the main contribution of the paper, we will describe and illustrate it in details. This section is concerned with modeling the voting setting and our chosen uncertainty model.

2.1 Modeling Plurality Elections from the Perspective of Strategic Voting

Let N be a set of voters where $N = \{1, \dots, n\}$, and M be a set of candidates where $M = \{1, \dots, m\}$. As we study strategic voting, we assume that $m > 2$, as per the Gibbard-Satterthwaite theorem [10, 30] voting rules are susceptible to manipulation only when there are at least three candidates. We assume that each voter $i \in N$ has some true underlying preferences over candidates represented by a linear order $\succ_i$ over candidates, but that those preferences can be imperfectly known. Let Π^m be the set of all possible preference orders for m candidates. In this paper, the winner of an election is determined by the Plurality voting rule where ties are broken lexicographically. Let $b_i \in M$ denote the ballot of voter i and $b \in M^n$ denote the ballot profile. The winner under Plurality of the ballot profile b is $\mathcal{W}(b) \in \arg\max_{x \in M} s_x(b)$, where $s_x(b) := |\{i \in N : b_i = x\}|$ and a lexicographic tie-breaking order, denoted by $\rhd$, is used if necessary. Given a number n of voters, we will denote by $\mathcal{S} = \{s : \sum_{x \in M} s_x(b) = n\}$ the space of possible score vectors.

By abuse of notation, we sometimes directly write $\mathcal{W}(s)$ to refer to the winner of a score vector s, and more generally use a ballot profile and its corresponding score interchangeably. Let b^T denote the truthful ballot profile, i.e., $b_i^T \succ_i x$ for every candidate $x \in M \setminus \{b_i^T\}$ and voter $i \in N$, and s^T denotes the candidates' scores in b^T.

Thanks to the Gibbard–Satterthwaite impossibility theorem, we know that strategic voting is unavoidable. This applies in particular to the voting rule considered here, namely plurality. Nevertheless, a voter can only strategize if she has some information about the current state of the election. In plurality elections, this is relatively straightforward, as knowing the current scores is sufficient to determine whether strategic voting is beneficial. However, one can easily argue that the broadcast score is subject to uncertainty, as polling institutes can only survey a subset of voters, and voters themselves may not be entirely certain about their preferences before the election day. Two models have met consensus in the literature to handle this problem.

2.2 Representing uncertain votes and polls

Existing representations Existing uncertainty representations typically focus on sets and probabilities. The main ones encountered in strategic voting literature are:

- sets $S \subseteq \mathcal{S}$ of possible scores, as for instance in the model of Meir [20]. This is arguably one of the simplest representations of uncertainty.
- probabilities $p(s)$ of scores over $\mathcal{S}$, which Myerson and Weber [27] derived from observed votes through a multinomial model. For our current contribution, it suffices to retain that it produces a probability over scores, and we will not detail the construction.
- The paper of Conitzer et al. [3] build on the model of Konczak et al. [16] using the notions of possible and necessary winners, considering that a voter uncertainty is given by a partial order over the M candidates, provided in the form of pairwise comparisons. As we are here considering plurality rules, this will be equivalent to consider the set of maximal elements of the partial as the potential vote ballot.

While our approach can cope with any subset of $\mathcal{S}$, in further examples, we will often adopt an interval-based notation for sets of scores, where an interval $[\underline{s}_x, \overline{s}^x]$ assigned to a given candidate represents the range within which their score may vary. This is a natural assumption that allows the neighborhood to be connected. This is illustrated in the following example:

Example 1. *If $n = 5$ and $|M| = 3$ with $M = \{a, b, c\}$, the set-valued score $S = ([2, 3], [1, 3], [0, 2])$ means that the score of b will be between 1 (one voter will vote b for sure) and 3 (up to three voters may vote b). Such a result based on sets, for instance, of five voters having given as sets of possible ballots $\{a\}, \{a\}, \{a, b, c\}, \{b\}, \{b, c\}$, with the last vote resulting for instance from the partial preference $b \succ_i a$.*

Let us recall that whenever we use this interval-valued representations, the sum of the scores $s = (s_1, \dots, s_m)$ picked within $[\underline{s}_i, \overline{s}^i]$ should still sum up to n, as it is a subset of $\mathcal{S}$.

Probability sets We propose to consider in this paper that uncertainty on votes is given by *convex probability set* $\mathcal{P}$ containing all measures P over the space of possible score vectors $\mathcal{S}$, either derived from poll results or voters uncertainty. The next example describe a very simple situation that this model can capture but that sets or probabilities cannot.

Example 2. *Assume we have two voters and two candidates $\{a, b\}$[1]. Each voter state that she is more willing to vote for a rather than b, while being unsure. This can be translated by $\mathcal{P}_i = \{p_i \mid p_i(\{a\}) \geq p_i(\{b\})\}$ for voter i. This info cannot be faithfully captured by a set nor by a precise probability.*

Assuming voters are independent, these uncertain votes then induce a set of possible probabilities over $\mathcal{S}$, with $P(\{(2, 0)\}) = p_1(\{a\})p_2(\{a\})$, $P(\{(1, 1)\}) = p_1(\{a\})p_2(\{b\}) + p_1(\{b\})p_2(\{a\})$, $P(\{(0, 2)\}) = p_1(\{b\})p_2(\{b\})$. Given the constraints on $\mathcal{P}_i$, one can then deduce that $P(\{(2, 0)\}) \geq P(\{(0, 2)\})$ and $P(\{(1, 1)\}) \geq P(\{(0, 2)\})$, hence that a will most likely be elected, but not whether $P(\{(2, 0)\}) \geq P(\{(1, 1)\})$, meaning that the most probable score is not known for sure.

This set can be interpreted as a collection of plausible probability distributions. Rather than observing a fixed set of scores or a single probability around the broadcast score, voter uncertainty is modeled through the probability set $\mathcal{P}$. This very generic model includes the previously cited one:

[1] The example is minimalist and we will provide examples with more than two candidates when considering strategic voting.

- a **set** S of scores, the model retained by Meir [20] and the one derived from incomplete preferences in case of plurality Conitzer [3] is captured by the probability set defined as $\mathcal{P}_S = \{P : P(S) = 1\}$, summarised by the constraint $\underline{P}(S) = 1$. In particular, if S is a neighborhood of the true score s with respect to a distance d that we leave unspecified at this stage, then we recover the uncertainty representation of Meir [20].
- a **probability** p over $\mathcal{S}$ simply amounts to considering $\mathcal{P} = \{p\}$, in which case our set reduces to a singleton. This means that any framework resulting in such a probability can be easily captured by our setting, as is the case for Myerson [27] (we will not detail here how the probability is obtained by Myerson [27], as we only deal with the problem of making a strategic move once this probability is obtained, we can theoretically consider any probability defined on $\mathcal{S}$).

Of course, manipulating generic sets of probabilities, whether one directly gives a set $\mathcal{P}$ on $\mathcal{S}$ or builds it from individual voter uncertainties $\mathcal{P}_i$, gives us a very expressive framework to describe uncertainties, but is not very computationally friendly nor very operational. This is why we now detail some specific models of interest [7], focusing especially on belief functions and their special cases.

Remark 1. *We restrict our uncertainty models to belief functions as they offer practical advantages, but some natural assessments would need to opt for the richer language of generic convex sets of probabilities, such as providing partial order between the probabilities of voting for some candidates [25]. Example 2 is of this kind. This also explains why we prefer to embed our approach in a very general framework.*

Belief functions A belief function over $\mathcal{S}$ consists in defining a positive mass function $\mathcal{M} : \mathcal{S} \to [0, 1]$ that sums up to one, i.e., $\sum_{S \subseteq \mathcal{S}, S \neq \emptyset} m(S) = 1$. From such a mass $\mathcal{M}$, we define two bounds over events $A \subseteq \mathcal{S}$ that are defined as

$$\underline{P}(A) = \sum_{S \subseteq A} m(S), \quad \overline{P}(A) = \sum_{S \cap A \neq \emptyset} m(S) \tag{1}$$

and from which one can define a corresponding set of possible probabilities

$$\mathcal{P}_{\mathcal{M}} = \{P : \forall A, \underline{P}(A) \leq P(A) \leq \overline{P}(A)\} \tag{2}$$

Belief functions are strictly less expressive than generic sets $\mathcal{P}$, as can be seen from the fact that $\mathcal{P}_{\mathcal{M}}$ is induced by constraints on events alone. For instance, the set $\mathcal{P}$ from Example 2 cannot be exactly captured by belief functions. However, they are still quite expressive, and notably include probabilities and sets as special cases: probabilities correspond to masses bearing only on singletons, while a set E correspond to the mass $m(E) = 1$.

Example 3. *Consider a belief function over the set of possible scores $\mathcal{S}$. For example, one can define:*

$$\mathcal{M}(\{(1, 1, 1), (0, 2, 1)\}) = \mathcal{M}(([0, 1], [1, 2], 1)) = 1$$

This mass function, that corresponds to a set, describes a situation where two voters are certain to vote for b and c, respectively, while one voter is hesitating between a and b.

Remark 2. *Another possible way to model the problem is to represent the uncertainty over each voter's ballot using a mass function $\mathcal{M}_i$ for voter i defined over M. This is what has been done to recover Conitzer's model [3]. From those, one can easily build a joint mass function over M^n (hence over $\mathcal{S}$) : if E_i are some subsets of M receiving positive masses for voter i, then the joint mass given over $E_1 \times \ldots \times E_n$ is simply $\mathcal{M}(E_1 \times \ldots \times E_n) = \prod_{i=1}^{N} \mathcal{M}_i(E_i)$, which is well-suited to denote independence between agents [31]. Each imprecise ballot can then be mapped to a corresponding set of possible score vectors. Therefore, for the remainder of the paper, we assume a modeling directly on the score space, that is, a mass function $\mathcal{M} : \mathcal{S} \to [0, 1]$.*

A key concept in the literature on belief functions that we will use later on is the *pignistic probability* p^* of $\mathcal{M}$, which selects a single probability from $\mathcal{P}_\mathcal{M}$. It is defined as

$$p^*(s) = \sum_{S \ni s} \frac{m(S)}{|S|},$$

and corresponds to the Shapley value of $\underline{P}(A)$ viewed as a game, also reflecting a Laplacian assumption within each focal set S. It can also be interpreted as some vertex center of gravity of $\mathcal{P}_\mathcal{M}$, and thus as a meaningful precise representative of $\mathcal{P}_\mathcal{M}$ (just as a center of gravity meaningfully represents a solid), if one must be selected. Also, if the mass $\mathcal{M}$ is already a probability distribution p, then $p^* = p$.

As mentioned earlier, belief functions are quite useful to model uncertain polls [17], especially if we allow voters to express their opinion as sets of possible candidates, just as in the work of Conitzer et al. [3] once incomplete preferences are transformed into possible ballots.

Example 4. *Consider the case of single-peaked preferences [2] where candidates* $\{a, b, c, d, e\}$ *are ordered with a left-right axis. One could then naturally weaken the result of a poll by considering that each voter could also vote for the two nearest candidates on each side (votes for a become* $\{a, b\}$*, for b become* $\{a, b, c\}$*, etc.). Imagine that an initial poll of 100 voters gives the initial score* $(10, 30, 10, 30, 20)$*. With a small abuse of notation, we have the following mass* $\mathcal{M}(\{a, b\}) = 0.1$*,* $\mathcal{M}(\{a, b, c\}) = 0.3$*,* $\mathcal{M}(\{b, c, d\}) = 0.1$*,* $\mathcal{M}(\{c, d, e\}) = 0.3$*,* $\mathcal{M}(\{d, e\}) = 0.2$*, from which we can derive a mass function on possible scores. Note that, by a slight abuse of notation, we write the mass function over candidates, since under the plurality rule this does not change the available information. The translation from one representation to the other is straightforward, and this notation is sometimes more convenient.*

Finally and before stepping to the decision part of our model, we will mention and illustrate two particular cases of belief functions that are of interest to us and constitute the framework in which we will place our convergence results of Section 4.

Necessity measures A necessity measure is a belief function where the masses are given to nested elements, that is $m(S_i) \neq 0$ and $m(S_j) \neq 0$ iff $S_i \subset S_j$ or $S_j \subset S_i$.

Example 5. *Consider a voter i providing the following information about* $\{a, b, c\}$*: "I will not vote for c, and hesitate between a and b, with higher chances to vote for a". Such an opinion could be modelled by the mass function* $\mathcal{M}^i(\{a\}) = 0.5$ *and* $\mathcal{M}^i(\{a, b\}) = 0.5$*, meaning that* $[\underline{P}(\{a\}), \overline{P}(\{a\})] = [0.5, 1]$ *and* $[\underline{P}(\{b\}), \overline{P}(\{b\})] = [0, 0.5]$.

Example 6. *Consider three candidates* $\{a, b, c\}$ *and the initial poll giving* $s^* = (0, 2, 1)$ *as a result on 3 voters. It would be natural to consider that* s^* *is the most likely, fully plausible result, but that nearby results are possible as well, albeit less as they are further away from* s^**. Building neighbourhood* S_r^* *by considering r the maximal number of voters that change their votes (so that the sum of votes remains the same), we get* $S_1^* = \{s^*\} \cup \{(0, 2, 1)(1, 1, 1), (0, 1, 2), (1, 2, 0)\}$ *and* $S_2^* = S_1^* \cup \{(2, 0, 1)(1, 0, 2), (0, 0, 3), (2, 1, 0)\}$*. One could then consider the mass* $\mathcal{M}(\{s^*\}) = \alpha$*,* $\mathcal{M}(S_1^*) = \beta$ *and* $\mathcal{M}(S_2^*) = 1 - \alpha - \beta$ *with* $\alpha \geq \beta \geq (1 - \alpha - \beta)$ *to denote a decreasing trust of capturing the true results as we get further away from* s^**.*

These examples illustrate the kind of model we will consider to extend Meir et al.'s results [20] convergence results in Section 4.

Inner measures An inner measure is a belief function [5, Ch. 2.], where masses are given on a partition of the space, that is $m(S_i) \neq 0$ and $m(S_j) \neq 0$ if $S_i \cap S_j = 0$ and $\cup_{S, m(S) > 0} S = \mathcal{S}$ (or a subset of interest).

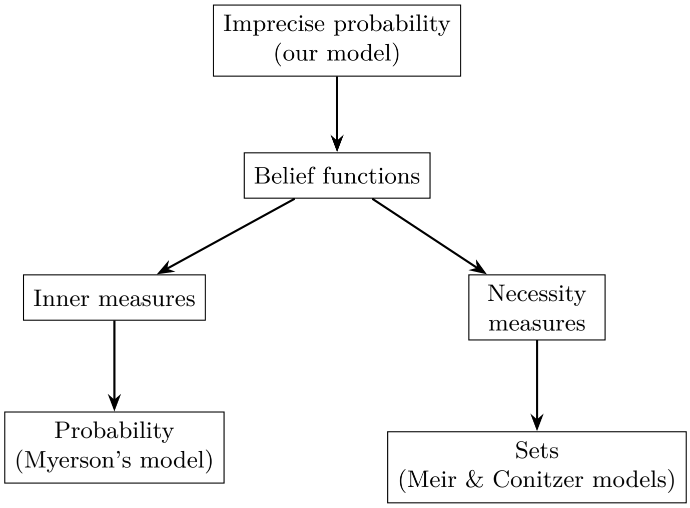

Figure 1: Overview of Our Uncertainty Model and Its Special Cases

Finding examples of inner measures naturally arising in a voting scheme is arguably more difficult, yet while necessity measures are well-suited to equip set-based approaches with gradual notions of enlargement (due to the nestedness), inner measures are well-suited to extend probabilistic models, as they rely on a partition. They are therefore useful to consider theoretically, as they may provide further insights about probabilistic models. This is reflected in Figure 1.

2.3 Relations to existing models

Meir's model In his work, Meir defines a neighborhood of scores with respect to a distance d : $S_r(s) = \{s' \in \mathcal{S} \mid d(s, s') \leqslant r\}$, d can be for instance some ℓ_p (Meir typically takes the ℓ_1 [20]). However, Meir [20] requires one to fix r and to consider that any score of the neighborhood of scores is considered identically. Our models can circumvent this limit by considering multiple r values and by decreasing the plausibility of successive sets, just like in Example 6.

Example 7. *Consider three candidates* $\{a, b, c\}$*,* $n = 30$ *the number of voters and the broadcast score being* $s = (10, 9, 11)$*. We will define beliefs on the neighborhood of scores directly on* $\mathcal{S}$*,* $\mathcal{M}(S_1) = 0.5$*,* $\mathcal{M}(S_2) = 0.3$ *and* $\mathcal{M}(S_3) = 0.2$ *with* $S_1 \subset S_2 \subset S_3$*, hence this particular structure is also a necessity measure.*

This model describes exactly what we expect of a belief function that models a certainty decreasing with respect to the distance to the broadcast poll s. They are therefore the ideal candidate to extend Meir's approach [20].

Remark 3. *One similar approach is proposed by Lev et al. [18]. They consider a criterion according to which, whenever there exists a locally dominating move at some level, the voter changes her vote. A key difference with our approach is that we can naturally embed multiple neighbourhoods chosen in advance in a single decision rule, as we shall see in the next section.*

We may also want to consider the same notion of neighbourhood but to have a model with a more probabilistic flavour. In this case, one can consider the partition generated by the S_i sets, and associate a probability mass to each member of this partition, generating a so-called inner measure [5, Ch. 2.] on any event of the initial space $\mathcal{S}$. An intrinsic interest of this model is that it remains a probabilistic one, even if defined on an algebra where the atoms are non-singleton sets of scores.

Example 8. *Consider three candidates* $\{a, b, c\}$*,* $n = 30$ *the number of voters and the broadcast score being* $s = (10, 9, 11)$*. We will define uncertain information directly on* $\mathcal{S}$*,* $\mathcal{M}(S_1) = 0.5$*,* $\mathcal{M}(S_2 \setminus S_1) = 0.3$ *and* $\mathcal{M}(S_3 \setminus S_2) = 0.2$*.*

Conitzer's model Incomplete preferences of voters are also naturally embedded in our approach. Indeed, as each incomplete preference gives rise to an imprecise ballot (the maximal elements of the partial order induced by the incomplete preference), one can easily map them to a set of ballots, hence to a belief function. The next example illustrates that.

Example 9. *Consider three candidates* $\{a, b, c\}$*,* $n = 3$ *the number of voters and the following preferences given by voters:*

$$b \succ_1 c \succ_1 a$$

$$c \succ_2 a \succ_2 b$$

$$a \succ_3 c$$

where voter 3 may only know that she prefers a to c, while the full completion remains unknown. For instance, the completion could be $a \succ c \succ b$*,* $b \succ a \succ c$*, or* $a \succ b \succ c$*. In all these cases, c is never ranked first, whereas the top-ranked candidate may be either a or b. This is naturally modelled by the belief function over* $\mathcal{S}$ *defined as* $\mathcal{M}(\{(1,1,1),(0,2,1)\}) = \mathcal{M}(([0,1],[1,2],1)) = 1$*.*

Remark 4. *In larger elections, it is natural to generalize this kind of reasoning as a subset of voters may be certain to vote for a given candidate, while another subset may still hesitate between two (or more) alternatives.*

Let us now proceed to the decision part of our model, describing how one voter may decide to switch its votes for another candidate.

3 Single-step decision model

We now introduce our decision model, illustrate it and link it with the previous single-step manipulation works we mentioned previously.

3.1 The decision model

From a probability set $\mathcal{P}$ on $\mathcal{S}$ and some utility $u : \mathcal{S} \to \mathbb{R}$, we define the lower expectation

$$\underline{\mathbb{E}}_{\mathcal{P}}(u) = \inf_{P \in \mathcal{P}} \mathbb{E}_P(u) \tag{3}$$

where $\mathbb{E}_P$ is the standard, linear expectation operator. Upper expectation $\overline{\mathbb{E}}_{\mathcal{P}}(u)$ can be defined likewise, taking a sup over $\mathcal{P}$. Lower and upper expectations form the basis of decision making under uncertainty when considering a probability set $\mathcal{P}$, and we will use them to model strategic decisions and manipulation taken by a given voter. These lower and upper expectations should be interpreted from the voter's perspective as the worst and best expected scenarios regarding the voting situation, based on their perception of uncertainty and their individual preferences.

Within the specific cases we have already mentioned in Section 2, these lower and upper expectations take convenient closed analytical form:

- In the case of a set $S \subseteq \mathcal{S}$ of scores, Equation (3) has for solution $\underline{\mathbb{E}}_{\mathcal{P}_S}(u) = \inf_{s \in S} u(s)$, something straightforward to compute ($\overline{\mathbb{E}}$ is obtained by taking a sup);
- In the case of a single probability $\mathcal{P} = \{p\}$, we just have that $\underline{\mathbb{E}}_{\mathcal{P}}(u) = \overline{\mathbb{E}}_{\mathcal{P}}(u)$ reduce to classical expectation;
- For belief functions, Equation (3) as well as the upper expectation can be easily solved by adopting the following formula using $\mathcal{M}$, i.e.,

$$\underline{\mathbb{E}}_{\mathcal{M}}(u) = \sum_{S \subseteq \mathcal{S}} m(S) \inf_{x \in S} u(x), \tag{4}$$

$$\overline{\mathbb{E}}_{\mathcal{M}}(u) = \sum_{S \subseteq \mathcal{S}} m(S) \sup_{x \in S} u(x). \tag{5}$$

In this model, we want to encode the strategic behavior of voters under uncertainty. To do so, let us define a function $st(s, a_i, a_i') : \mathcal{S} \times M^2 \to \mathcal{S}$ that sends back an updated score if voter i moves his vote from a_i to a_i'. To simplify the reading of equations using it, we will use the shorthand $s_{a_i \to a_i'} := st(s, a_i, a_i')$ for the update of score s after a strategic move. Then, let $u_i : M \to \mathbb{R}$ be for each voter i the associated utility that reflects how satisfied the voter is with the current winner $\mathcal{W}(s)$. This utility function could be more complex, e.g., be defined on M^2 to denote pairwise preferences over candidates and represent how much some candidates are preferred to other ones, however we will stick to this simpler definition depending only on M, as it is sufficient for our purpose. Also note that by Debreu et al. [4], we can always build a utility function from the strict linear orders $\succ_i$. Therefore, to understand how voters may change their vote, we need to evaluate the benefit of moving from a_i to a_i' that we denote

$$u_i(a_i'|a_i, s), \tag{6}$$

that is the utility of moving to a_i', given the initial state of affairs s and the current vote a_i. In particular, if we are certain about our current state of affairs and we know the score vector s, it suffices for (6) to be positive for the voter to make a strategic move, that is

$$a_i' \succeq a_i \text{ iff } u_i(a_i'|a_i, s) \geq 0$$

where $\succeq$ denotes here that action of voting for a_i' is preferable to voting for a_i.

However, in our model this is not the case since agents receive a probability set $\mathcal{P}$. Assuming that we have some probability set $\mathcal{P}$ defined over $\mathcal{S}$, we will define four decision criterion (named DC) under uncertainty.

- The first criteria that we will consider is a pessimistic criteria to decide whether an action is better than another as follows:

$$a_i' \succ_{pess} a_i \text{ iff } \underline{\mathbb{E}}_{\mathcal{P}}(u_i(a_i'|a_i, \cdot)) \geq 0$$
$$\text{and } \overline{\mathbb{E}}_{\mathcal{P}}(u_i(a_i'|a_i, \cdot)) > 0$$

  where uncertain scores knowledge is modeled by $\mathcal{P}$. In essence, this corresponds to the maximmin criterion put forward by Gilboa et al. [11].

- A pignistic criteria to decide whether an action is better than another as follows:

$$a_i' \succeq_{pig} a_i \text{ iff } \mathbb{E}_{p^*}(u_i(a_i'|a_i, \cdot)) \geq 0,$$

  where p^* is the pignistic probability. Note that it includes standard probabilistic decision as a special case.

- A mixture criteria to decide whether an action is better than another as follows:

$$a_i' \succeq_{\alpha, p^*} a_i$$
$$\text{iff } \alpha \cdot \underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i, \cdot))$$
$$+(1-\alpha) \cdot \mathbb{E}_{p^*}(u_i(a_i'|a_i, \cdot)) \geq 0$$

  with $\alpha \in [0, 1]$. Note that α can be seen as a level of completeness of the obtained preferences between possible moves, as it allows one to go from a rather partial order ($\alpha = 0$) to a complete one ($\alpha = 1$).

- Another mixture of criteria to decide whether an action is better than another as follows:

$$a_i' \succeq_{H(\alpha)} a_i$$
$$\text{iff } \alpha \cdot \underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot))$$
$$+(1-\alpha)\cdot \overline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot)) \geq 0$$

with $\alpha \in [0,1]$. In essence, this corresponds to the well-known Hurwicz criterion [14] that intends to balance between optimism ($\overline{\mathbb{E}}$) and pessimism ($\underline{\mathbb{E}}$), and that has been justified within a belief function setting by Denoeux et al [6].

Note that for the three last rules, the strict relation $\succ$ corresponds to strict positive values on the right side. Those decision rules are akin to commonly used decision rules within the imprecise probabilistic setting [33].

To sum up, we describe an election under plurality with a lexicographic tie breaking and strategic voting under uncertainties with the tuple $(N, M, \succ, \rhd, \mathcal{P}, (u_i)_{i\in N}, DC)$. Let us give an example putting our framework at work, where the considered uncertainty is neither a set nor a probability.

Example 10. *Consider candidates $\{a,b,c\}$ and $n=3$, and consider the situation where the first voter is as described in Example 5, while the second and the third voters are certain that their ballots are b and c. In this case, the belief function over $\mathcal{S}$ is defined as $\mathcal{M}(\{(1,1,1)\}) = 0.5, \mathcal{M}(([0,1],[1,2],1)) = 0.5$. Let us now illustrate one of our decision rules on this same example, to give the reader an idea of how this plays out.*

Consider for instance that the preference of the first voter is given by $b \succ_1 c \succ_1 a$, which is coherent with its ballot and a Hurwicz criterion with $\alpha = \frac{1}{3}$, meaning that the weight of optimism is larger. We will consider the following utilities

$$\forall i \in N, u_i(a_i'|a_i,s) = \begin{cases} 1 & \textit{if } \mathcal{W}(s_{a_i\to a_i'}) \succ_i \mathcal{W}(s), \\ 0 & \textit{if } \mathcal{W}(s_{a_i\to a_i'}) \sim_i \mathcal{W}(s), \\ -1 & \textit{else.} \end{cases}$$

We want to evaluate the deviation from b to c. Then, with the linearity of the mass functions, we compute

$$\underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot)) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0$$

and

$$\overline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot)) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$$

Indeed, for the precise set $\{(1,1,1)\}$ with mass 0.5, the move makes c elected instead of a, and the utility is a constant 1 (there is no uncertainty on set $\{(1,1,1)\}$). For the set $([0,1],[1,2],1)$ that also has mass 0.5, the move is still beneficial for the possible score $(1,1,1)$, but would bring negative utility for $(0,2,1)$, as c would be elected instead of b, a clear downside if the voter preferences are $b \succ_1 c \succ_1 a$. Thus, we get:

$$\frac{1}{3}\cdot \underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot))$$
$$+(1-\frac{1}{3})\cdot \overline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i,\cdot))$$
$$= \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 1 = \frac{2}{3} \geq 0.$$

The conclusion is that voter 2 will do the deviation from b to c with such a decision criterion.

All this gives us a very rich uncertainty modeling and decision framework, in which one can easily separate what comes from the uncertainty model, and what comes from the selected decision rule. Let us now see how our approaches connect with the existing literature.

3.2 Relations with Existing Models

We now discuss that the three models from Meir et al. [23, 20], Myerson et al. [27] and Conitzer et al. [3] are specific case of our approach, demonstrating its expressiveness.

For the first one, we consider the model $S_r(s)$ mentioned in Section 2.3. By abuse of notation, we will use S_r when it is clear from the context. The seminal work of Meir [23] is working on the special case $r = 0$, meaning the poll information is complete for plurality and certain. This corresponds to considering voters $i \in N, \mu_i(S_0) = 1$, particular utility functions:

$$\forall i \in N, u_i(a'_i|a_i, s) = \begin{cases} 1 & \text{if } a'_i = \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s), \\ 0 & \text{if } a'_i \neq \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s), \\ 0 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \sim_i \mathcal{W}(s), \\ -1 & \text{else,} \end{cases}$$

and the pessimistic decision rule $\succeq_{pess}$ to retrieve Meir's model [23] as a particular case. Recall that in [20], the convergence is guaranteed and we call these strategic moves "direct best response" since all strategic moves where a voter deviates are to the new winner.

Meir [24] then extended this framework to add uncertainty by considering some strictly positive value $r_i > 0$ for voter i. This comes down in our model to take, $\forall i \in N, \mu_i(S_{r_i}) = 1$, particular utility functions

$$\forall i \in N, u_i(a'_i|a_i, s) = \begin{cases} 1 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s), \\ 0 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \sim_i \mathcal{W}(s), \\ -1 & \text{else.} \end{cases}$$

and again the pessimistic decision rule $\succeq_{pess}$ to recover the model.

For the probabilistic model of Myerson et al. [27], they derive from observed votes a weight vector $q = (q_1, \dots, q_m)$, and the probability over score vectors $\mathbb{P}(s)$ follows a multinomial distribution $s \sim M(n, q)$. The utilities can be chosen arbitrarily provided they satisfy the voter's preferences:

$$u_i(a'_i|a_i, s) \geq 0 \text{ iff } \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s).$$

Since in the probabilistic case all decision rules we have described in Section 3.1 reduce to a classical expectation operator, there is no need to specify a specific behaviour.

The last model we want to encompass is the one from [3] with incomplete preferences. As we already showed in Section 2, the incompleteness on each voter's preference can easily be summarized as an uncertainty set on scores that we call S and where $\mu(S) = 1$. It remains to choose the decision rule with respect to this set. Conitzer et al.[3] adopt a pessimistic decision criterion, which amounts to using the following utilities:

$$\forall i \in N, u_i(a'_i|a_i, s) = \begin{cases} 1 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s), \\ 0 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \sim_i \mathcal{W}(s), \\ -1 & \text{else.} \end{cases}$$

and the pessimistic decision rule $\succeq_{pess}$.

3.3 A word on complexity

Let us briefly situate our model with respect to the computational complexity of computing a manipulation. Prior work has already investigated related questions, and it is useful to distinguish two broad cases. For plurality, when one must evaluate each scenario induced by uncertainty, for instance, under incomplete preferences, this essentially amounts to reasoning over all completions. This is closely related to the possible winner task and is #P-hard [3]. In particular, under criterion that aggregate across completions (e.g., computing an expectation under a pignistic rule), one cannot avoid considering the full set of completions. By contrast, when the decision criterion is simpler, such as a pessimistic (worst-case) criterion, plurality requires only limited information; it suffices to identify the winner in the relevant worst-case scenario. In that case, the problem can be rewritten as a flow problem as in Conitzer et al. [3] which can be solved in polynomial time. We may also mention other decision criterion that remain simple enough to be computed in polynomial-time [8], such as the optimistic criterion (i.e., the preferred candidate wins in at least one completion) and opportunistic manipulation (i.e., the preferred candidate wins in every completion in which some manipulation is feasible).

The same dichotomy can be applied to our approaches: if a decision rule requires enumerating all possible completions of a vote, we will end up with #P-hard problems. This is the case for decision rules involving the pignistic probability, for instance. On the other hand, and provided the belief function mass $\mathcal{M}$ is positive only on a polynomial number of sets, rules that only require identifying one completion remains polynomial.

In our general model, including these cases as particular cases, we then know that under any criteria that needs to compute every possible scenario then the decision will be also #P-hard to decide, or worse if we consider quite generic/complex models. However, the case of a pessimistic decision criterion remains solvable in polynomial time as long as the number of sets receiving positive mass (in the case of belief functions) remains polynomial, essentially because it does not require computing probabilities or weights. Thanks to this observation, a standard maximum-flow approach [3] can be applied to obtain a polynomial-time algorithm for manipulation with the pessimistic criterion.

This also opens up an interesting avenue of research: given that in the probabilistic case all decision rules amount to classical expected value, could we weaken the initial probabilistic information, e.g., by considering an inner measure on a coarse partition of linear orders, and take a tractable decision rule (pessimistic and optimistic) in order to approximate intractable probabilistic manipulation rules? To be more precise, if computing an expectation $\mathbb{E}(\cdot)$ for a probability is intractable, can we find a set $\mathcal{P}$ containing p such that computing $\underline{\mathbb{E}}(\cdot), \overline{\mathbb{E}}(\cdot)$ is tractable, and where $\underline{\mathbb{E}}(\cdot)$, $\overline{\mathbb{E}}(\cdot)$ are reasonably close to each other? This would let us reach a decision in some cases, and in others, quantify the remaining uncertainty. Note that such a question is made possible only by the fact that our models unify set-based with probabilistic-based decision (respectively known as decision under uncertainty, and decision under risk [12]) in a single framework, thereby allowing one to easily transfer notions used in one setting to the other setting.

Now that we have a fully fledged decision framework allowing us to model whether or not a voter will proceed to a strategic move or manipulation, we study the multi-step case, in particular showing that classical convergence results can be reinterpreted and extended to specific instantiations of our framework.

4 Multi-step Manipulations

In this section we generalize existing results on convergence from Meir [20] by adding quantitative uncertainty about the score as in Example 7 and Example 8. Let us recall that a neighborhood of scores S_r with respect to a distance d is defined as in Meir's model [20]: $S_r(s) = \{s' \in \mathcal{S} \mid d(s, s') \leqslant r\}$, where d can be ℓ_1, ℓ_∞, earth moving distance (EMD) or another one. Let us consider the ℓ_1 distance, which is easy to interpret in terms of adding or removing votes, without necessarily preserving the total number of votes, which may vary by one. This behavior can be easily explained by considering a voter who chooses to abstain. For each voter i, we will let r_i be the support of their uncertainty, so that S_{r_i} is the biggest set of voter i's uncertainty.

We now formalize the notion of equilibrium in this model through the following definition.

Definition 1. *An equilibrium is a situation where no voter has an incentive to deviate from its ballot, i.e.* $\forall i \in N, \nexists\ a'_i$ *such that* $a'_i \succ_{DC} a_i$. *We assume voters will only make a strategic move if their preferences are strict.*

4.1 Extending the Meir Framework

Let us consider a belief function for voter i, $\mathcal{M}^i : \mathcal{S} \to [0,1]$ that sums up to one, i.e., $\sum_{S \subseteq S_{r_i}, S \neq \emptyset} m^i(S) = 1$. This notation for belief functions indexed by voters should not be confused with that of Remark 2, where the question was to model each ballot independently, while here we consider that the scores come from a unique poll, but that each voter may be more or less skeptical about its result. We will consider two particular cases of interest. The first is the case of nested sets, i.e. $\forall i \in N, \forall k \geqslant 1, m^i(S_k) = \beta^i_k$, with $\sum_{k=1}^{r_i} \beta^i_k = 1$. Indeed, we might want to describe the fact that the belief mass is depending on the distance to the true score. Second, we look at the case of *partitioned* belief function, i.e. $\forall i \in N, \forall k \geqslant 2, m^i(S_k \setminus S_{k-1}) = \beta^i_k$, with $\sum_{k=1}^{r_i} \beta^i_k = 1$ and $\forall i \in N, m^i(S_1) = \beta^i_1$. In both cases, we will assume decreasing $(\beta^i_k)_{1 \leqslant k \leqslant r_i}$, meaning that our evidence decreases as we get further from the observed s. For the rest of this section, we consider the following utilities:

$$\forall i \in N, u_i(a'_i|a_i, s) = \begin{cases} 1 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \succ_i \mathcal{W}(s), \\ 0 & \text{if } \mathcal{W}(s_{a_i \to a'_i}) \sim_i \mathcal{W}(s), \\ -1 & \text{else.} \end{cases}$$

Theorem 1. *Voters considering uncertainty given by a nested or partitioned belief function, and making strategic decisions according to either pessimistic* ($\succeq_{pess}$) *or mixed* ($\succeq \check{\mathrm{r}}_{\alpha,p^*}, \succeq_{H(\alpha)}$) *decision rules with* α *large enough, will converge to an equilibrium.*

Proof. At first, we will take $\forall i \in N, \forall k \geqslant 1, \mu_i(S_k) = \beta^i_k$, with $\sum_{k=1}^{r_i} \beta^i_k = 1$. We consider a feasible move $a_i \to a_{i'}$. We remark that the hypothesis that $\underline{\mathbb{E}}_{\mathcal{M}}(u_i(a'_i|a_i, s)) \geqslant 0$ implies that either:

- $\forall s \in S_{r_i}, u_i(a'_i|a_i, s) \neq -1$,
- $\exists \tilde{r} \in [0, r_i], \forall s \in S_{\tilde{r}}, u_i(a'_i|a_i, s) = 1$

since, if some neighborhoods have a negative utility, this must be compensated by a positive contribution, which happens only if all moves within a neighborhood the results. However, the second case is impossible, for the reason that if we allow a voter to transfer its vote to another candidate ($r = 1$), then there is a situation for which the voter is not pivotal, and therefore $\exists s \in S_{\tilde{1}}, u_i(a'_i|a_i, s) = 0$, showing that the second case never happens. Therefore, if

$$\underline{\mathbb{E}}_{\mathcal{M}}(u_i(a'_i|a_i, s)) \geqslant 0$$

this means that $\forall s \in S_{r_i}, u_i(a'_i|a_i, s) \in \{0, 1\}$, with at least one s giving the null value.

Second, note that we cannot have $\underline{\mathbb{E}}_{\mathcal{M}}(u_i(a'_i|a_i, s)) > 0$ if $\forall s \in S_{r_i}, u_i(a'_i|a_i, s) = 0$, and there must be a situation for which making this strategic move is a local dominance move, meaning it verifies Theorem 4 from Meir [20].

We do exactly the same reasoning for the second type of belief functions, i.e., $\forall i \in N, \forall k \geqslant 2, \mu_i(S_k \setminus S_{k-1}) = \beta^i_k$, with $\sum_{k=1}^{r_i} \beta^i_k = 1$ and $\forall i \in N, \mu_i(S_1) = \beta^i_1$.

For the mixed decision, one can prove that the decision behaves as the pessimistic one if α is large enough. □

We now want to go a step further by showing the convergence can hold even with a less, but still pessimistic behavior, namely the Hurwicz criterion with $\alpha > \frac{1}{2}$. This allows us to consider negative outcomes in the uncertainty neighborhood of scores. Let us consider belief functions as follows: $\forall i \in N, \forall k \geqslant 1, m^i(S_k) = \beta^i_k$, with $\sum_{k=1}^{r_i} \beta^i_k = 1$

Theorem 2. *Voters considering a belief function around the true score and making strategic votes according to a sufficiently pessimistic Hurwicz criterion (i.e. $\alpha > \frac{1}{2}$) will converge to an equilibrium.*

Proof. We assume that the weights β_k^i of our belief functions, defined as $\forall i \in N, \forall k \geqslant 1, \mu_i(S_k) = \beta_k^i$.

We consider a feasible move $a_i \to a_{i'}$. Using the fact that the lower (an upper) expectation is positive homogeneous, i.e., $\alpha\underline{\mathbb{E}}(f) = \underline{\mathbb{E}}(\alpha f)$, we get

$$\begin{aligned} &\alpha \cdot \underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i, s)) \\ &+ (1-\alpha) \cdot \overline{\mathbb{E}}_{\mathcal{M}}((u_i(a_i'|a_i, s)) \\ &= \sum_{i=1}^{r_i} \mu(S_i)[\alpha \inf_{s \in S}(u_i(a_i'|a_i, s)) \\ &+ (1-\alpha) \cdot \sup_{s \in S}(u_i(a_i'|a_i, s))] \end{aligned} \tag{7}$$

Let us denote by

$$\tilde{u_i}(S, a_i, a_i') = \alpha \inf_{s \in S}(u_i(a_i'|a_i, s))$$

$$+(1-\alpha) \cdot \sup_{s \in S}(u_i(a_i'|a_i, s))$$

the term associated to subset S, meaning that Equation (7) can be rewritten $\sum_{i=1}^{r_i} \mu(S_i)\tilde{u_i}(S, a_i, a_i')$, and that the strategic move is decided by a weighted average of $\tilde{u_i}$ values.

We distinguish six possible cases:

- Case A:
$$\exists s \in S, u_i(a_i'|a_i, s) = 1 \text{ and } \forall s \in S, u_i(a_i'|a_i, s) \geq 0$$
- Case B:
$$\exists s \in S, u_i(a_i'|a_i, s) = -1 \text{ and } \forall s \in S, u_i(a_i'|a_i, s) \leqslant 0$$
- Case C:
$$\exists s \in S, u_i(a_i'|a_i, s) = -1 \text{ and } \exists s \in S, u_i(a_i'|a_i, s) = 1$$
- Case D:
$$\forall s \in S, u_i(a_i'|a_i, s) = 0$$
- Case E:
$$\forall s \in S, u_i(a_i'|a_i, s) = 1$$
- Case F:
$$\forall s \in S, u_i(a_i'|a_i, s) = -1$$

However, cases E and F can never happen for a non-singleton S, for the same reasons as the ones advocated in the proof of Theorem 1. We then get

$$\tilde{u_i}(S, a_i, a_i') = \begin{cases} 1-\alpha & \text{Case A,} \\ -\alpha & \text{Case B,} \\ 1 - 2\cdot\alpha & \text{Case C,} \\ 0 & \text{Case D} \end{cases}$$

It is clear that $\alpha > \frac{1}{2}$ implies that $\tilde{u_i}(S, a_i, a_i') \geq 0$ in cases A only.

Therefore, if

$$\alpha \cdot \underline{\mathbb{E}}_{\mathcal{M}}(u_i(a_i'|a_i, s))$$

$$+(1-\alpha)\cdot \mathbb{E}_{\mathcal{M}}(u_i(a'_i|a_i,s)) \geq 0$$

then

$$\exists r'_i \leqslant r_i \quad \text{such that}$$

$$\exists s \in S_{r'_i}, u_i(a'_i|a_i,s) = 1$$

$$\text{and } \forall s \in S_{r'_i}, u_i(a'_i|a_i,s) \geq 0 \quad \text{(Case A)}$$

The last equivalence comes from the fact that only case A can lead to a positive $\tilde{u}_i(S, a_i, a'_i)$, and that the criterion is an average of such $\tilde{u}_i$ values. Therefore, it has to exist $r'_i \leqslant r_i$ such that $\tilde{u}_i(S_{r'_i}, a_i, a'_i) = 1-\alpha$. If $R_i^+ = \{j : |\tilde{u}_i(S_j, a_i, a'_i) = 1-\alpha\}$ is the set of neighborhood indices in case A, we need for $\sum_{j \in R_i^+} \beta_j^i$ to be large enough for the decision criterion to be positive. In other words, we can accept a move with some case C only if there exists a local dominance move in a smaller neighborhood that receives enough evidence. Using the result from section VIII in Meir [19] which tells us that the convergence holds for any level of uncertainty r_i and any starting point (even non-truthful states) for local dominance strategic behaviors, then we can do a bijection of our strategic moves and get the convergence also. □

4.2 The Case of Pignistic Probability

Let us now consider the case of a pignistic probability on a neighborhood of scores S_{r_i}, which is equivalent to having a uniform distribution $\mathcal{U}$ over all single scores within S_{r_i} for any voter i. Note that for the rest of this chapter the notation card is used for cardinality. If S_{r_i} is our single set with positive mass, the corresponding strategic behaviour is equivalent having the following criterion:

$$a'_i \succeq_{pig} a_i$$

$$\text{iff} \quad \mathbb{E}_{\mathcal{U}}(u_i(a'_i|a_i,s)) \geq 0$$

$$\text{iff} \quad \text{card}(s \in S \mid u_i(a'_i|a_i,s) = 1)$$

$$-\,\text{card}(s \in S \mid u_i(a'_i|a_i,s) = -1) \geq 0$$

When this criterion is not strict, it is obvious that we have a cycle because the same voter could move back and forth between two indistinguishable states. Therefore, we consider the strict version of it, namely

$$a'_i \succ_{\mathcal{U}} a_i$$

$$\text{iff} \quad \text{card}(s \in S \mid u_i(a'_i|a_i,s) = 1)$$

$$-\,\text{card}(s \in S \mid u_i(a'_i|a_i,s) = -1) > 0$$

Proposition 1. *With a uniform pignistic criterion on a neighborhood of scores of size 1 (with respect to the ℓ_1 distance), convergence is not guaranteed.*

Proof. Here is the counter example with a neighborhood of scores of size 1. Let us take the following profile:

$$\begin{array}{ccccccc}
a & \succ_1 & b & \succ_1 & c & \succ_1 & d \\
c & \succ_2 & a & \succ_2 & b & \succ_2 & d \\
d & \succ_3 & c & \succ_3 & a & \succ_3 & b \\
c & \succ_4 & d & \succ_4 & a & \succ_4 & b \\
a & \succ_5 & c & \succ_5 & d & \succ_5 & b \\
d & \succ_6 & b & \succ_6 & a & \succ_6 & c \\
c & \succ_7 & b & \succ_7 & d & \succ_7 & a \\
d & \succ_8 & b & \succ_8 & a & \succ_8 & c \\
b & \succ_9 & d & \succ_9 & c & \succ_9 & a \\
b & \succ_{10} & d & \succ_{10} & c & \succ_{10} & a
\end{array}$$

Voter 8 will move from d to a with a cardinal difference of 1. For clarity, let us detail the computation of this first move: the original score is $s = (2, 2, 3, 3)$, so

$$S_1 = \{(2,2,3,3),(1,2,3,3),(2,1,3,3),$$
$$(2,2,2,3),(2,2,3,2),(3,2,3,3),$$
$$(2,3,3,3),(2,2,4,3),(2,2,3,4)\}$$

that corresponds to add/remove one vote. When moving from d to a, Voter 8 is improved in the first state $(2, 2, 3, 3)$ as it becomes $(3, 2, 3, 2)$, and a is elected instead of c, which is better from voter 8's perspective. Voter 8 is also improved for states $(2, 1, 3, 3)$, $(2, 2, 3, 2)$ and $(2, 3, 3, 3)$, and deteriorated in states $(2, 2, 2, 3)$, $(2, 3, 3, 3)$ and $(2, 2, 3, 4)$. Other states are not impacted by this move. Then voter 2 will move from c to a with a cardinal difference of 2, voter 8 will move from a to d with a cardinal difference of 1 and finally voter 2 will move from a to c with a cardinal difference of 3. This creates a cycle, which prevents convergence. □

Of course, if the size of the neighborhood of scores is larger, there is not much hope for convergence either. We think this example is quite interesting, in particular when put in perspective with our result of Theorem 1 about partitioned belief functions. Indeed, Theorem 1 indicates that with the right algebra, it is possible to consider a probability measure and a corresponding decision rule such that convergence holds. However, this algebra coarser than the one induced by single score vectors. This indicates that the counter-example is not so much about having a probabilistic model itself than about the voters being perhaps too optimistic in their movement, hinting also at the fact that "optimistic" decision rules such as the Hurwicz criterion $\succeq_{H(\alpha)}$ with a too low pessimism index α, where one would rely mainly on the upper expectation are unlikely to lead to convergence of voting behaviours. Note that the distance here is the ℓ_1 distance, and that the number of voters does not sum to n. However, this can easily be interpreted as meaning that one voter does not participate, and is therefore still very reasonable. Moreover, similar examples can be found for other distances, including distances for which the number of voters sums to n.

5 Conclusion and Future Works

In this paper, we provide new tools to model uncertainty in strategic voting. We think that our work provides substantially new views on preference modeling in voting theory, and in particular in strategic voting models. In particular, we have shown that they capture standard settings in a unifying framework, and allow one to represent new uncertainty scenarios that were not captured by previous models. We have illustrated, through many examples, that this model can account for uncertainty arising both from polls and from the voters themselves. Moreover, we establish new convergence results in our framework that generalize existing ones, and we provide a key counterexample that point the limits of these convergence results.

This work opens up several promising research directions, both in strategic voting models and in preference modeling in voting theory more broadly. A first important question is how incorporating a more realistic model of uncertainty may influence election outcomes compared to classical iterative voting models [22]. Another avenue for future research is to design empirical studies aimed at identifying the parameters of our model, in order to represent the uncertainty present in both polls and voters' preferences more accurately in real-world elections. We also believe that the descriptive and expressive power of the framework is an advantage: it can formally capture more behaviours than other frameworks, and can ensure smooth transitions between set and probabilistic models. A challenge will however be to propose parametric models where the number of parameters does not explode. Suppose we are able to do so, it would allow us to reframe classical questions in iterative voting, such as whether equilibria can be computed efficiently [28], how an external agent might manipulate the outcome [1], or whether it increases social welfare [15]. One might also ask how such a model could be adapted to other voting rules. Finally, we believe this modeling can offer a new perspective on other voting problems, such as possible and necessary winners or elicitation of voters' preferences, by capturing uncertainty in a more quantitative flavor.

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