On Iterative Voting Outcomes in Plurality Elections

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1 Introduction

Strategic voting occurs when voters may have an incentive to misrepresent their true preferences to influence the election outcome. The Gibbard-Satterthwaite theorem [14, 26] establishes that no non-dictatorial and deterministic voting rule can fully prevent strategic manipulation. Therefore, some works have investigated how to make manipulation computationally difficult [1], or have identified preference domain restrictions in which manipulation is not beneficial (see, e.g., [24]). Another perspective to circumvent this impossibility result is, on the contrary, to allow manipulation and analyze its consequences - for example, by modeling voting as a strategic game and analyzing its outcomes.

Iterative voting [19] is a particular type of voting game in which voters are allowed to manipulate by performing successive strategic moves. Since voters manipulate sequentially, different possible outcomes can arise, depending on the order in which voters deviate. Our work examines the variability of outcomes due to this specific strategic behavior in concrete terms, by analyzing who can get elected and how the outcome may change.

In this article, we follow the classical initial model of [22], which is considered a standard baseline in iterative voting. In this model, the voting rule is plurality, and voters perform direct best responses when they are pivotal, i.e., they vote for the candidate they prefer the most among those they can directly make win by changing their vote. Under these assumptions, the iterative voting process is guaranteed to converge to an equilibrium. This raises a natural question: which candidates can actually be elected through such a process?

To address this question, we adapt the well-known notions of possible and necessary winners [18] to the iterative setting. More precisely, a possible iterative winner is a candidate for whom there exists a sequence of deviations eventually electing her at equilibrium. Analogously, a necessary iterative winner is elected in all possible equilibria that can arise from the iterative voting process.

For the necessary iterative winner problem, two configurations are possible: either several deviation sequences exist but they all lead to the same winner or, in a more extreme case, no voter can deviate from her truthful ballot, and thus the initial winner turns out to be the only iterative winner. To quantify how often this extreme scenario occurs, we analyze how frequently a truthful preference profile is already an equilibrium. We show that, under impartial (anonymous) cultures, the probability of this phenomenon admits a relatively high lower bound - and thus, so does the probability of the existence of a necessary iterative winner.

More generally, we investigate the computational complexity of determining whether a given candidate is a possible or necessary winner. Interestingly, it turns out that the problems fall into different complexity classes: while the possible iterative winner problem is NP-complete, the necessary iterative winner problem is polynomial-time solvable.

We then turn to a more qualitative property of iterative voting outcomes by analyzing how often the Condorcet winner, when one exists, is elected after the strategic process. Experimental results from [15] suggest that, under the impartial culture assumption, the Condorcet efficiency (i.e., the probability of electing the Condorcet winner when one exists) increases through iterative voting. We provide a formal proof of this phenomenon under both the impartial culture (IC) and impartial anonymous culture (IAC) assumptions.

Due to space restrictions, some proofs or parts of proofs are deferred to the supplementary material.

2 Related work

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In this article, we introduce the notion of possible and necessary iterative winners, which are adaptations of the well-known concepts of possible and necessary winners under incomplete preferences [18]. Up to our best knowledge, these notions have not been used so far to capture iterative voting outcomes. Nevertheless, in the context of manipulation in voting, they have been applied, e.g., to deal with incomplete information of the manipulators [7], as a list of intermediate results in an iterative elicitation process where voters can answer to the queries strategically [8], or to determine the outcome of sequential voting in the context of social networks [12]. Our computational results, stating a difference in complexity classes between the possible and necessary iterative winner problems, are consistent with the results of the literature regarding the initial notions. Notably, while the necessary winner problem under partial preferences is in P for all positional scoring rules, the possible one is NP-complete on this large class of rules except for the plurality and veto rules [2, 4, 18, 28].

Note that considering all possible iterative outcomes that can arise, depending on the sequence of voters' deviations, is similar in spirit to the notion of "parallel-universe" tie-breaking where the outcome is the set of all candidates who could win using a particular tie-breaking method. This has been particularly investigated for multi-stage voting rules where the choice of the candidates to eliminate at a given stage can highly impact the final winner of the voting procedure [10, 27]. The different sequences of eliminated candidates at the different stages can then be represented as a tree [10], and we can use a similar representation for all possible sequences of deviations potentially leading to different winners.

Instead of considering the diversity of iterative voting outcomes, where two equilibria are indistinguishable if they elect the same winner, one can focus more specifically on the possible equilibria that can be reached. This study has notably been conducted by [25], who establish that checking whether a given ballot profile is a reachable equilibrium is NP-hard, in a similar idea as our NP-completeness proof for the possible iterative winner problem.

In an orthogonal perspective, one can examine how good or bad are the outcomes of iterative voting. In particular, several works have analyzed the iterative voting outcomes by comparing them to the initial truthful one, following either a worst-case analysis based on an approach similar to the price of anarchy, or an average-case analysis [6, 16, 17]. Mostly, the outcomes have been evaluated via their social welfare, but it is also possible to consider other measures, such as the probability to elect the Condorcet winner when she exists [13]. [15] have followed this latter approach by experimentally analyzing the Condorcet efficiency of the iterative voting process. We go a bit further by theoretically demonstrate that indeed the iterative variant of plurality has a higher Condorcet efficiency compared to the initial plurality rule, where we consider the probability of electing a Condorcet winner over all possible deviation sequences with equal weights, under impartial (anonymous) cultures.

3 The Model

For any positive integer k, let $[k]$ define the set $\{1,\dots,k\}$. Let N be a set of voters where $N = [n]$, and M be a set of candidates where $M = [m]$. Each voter $i \in N$ has preferences over candidates represented by a linear order $\succ_i$ over candidates. Let $top(\succ_i)$ be the preferred candidate of i, i.e. $top(\succ_i) \succeq_i x$, for every $x \in M$. The set of all voters' preferences is called a preference profile and is denoted by $\succ$, i.e., $\succ := (\succ_1,\dots,\succ_n)$.

Let N^x be the set of voters who prefer x to any other candidate, i.e., $N^x := \{i \in N : top(\succ_i) = x\}$ and, for a given subset of voters $A \subseteq N$, let $A^{x\succ y}$ be the set of voters who prefer x to y, i.e., $A^{x\succ y} := \{i \in A : x \succ_i y\}$. A candidate x is the *Condorcet winner* if she beats all the other candidates in pairwise comparisons, i.e., $|N^{x\succ y}| > |N^{y\succ x}|$, for every candidate $y \in M \setminus \{x\}$. A *weak* Condorcet winner x is such that $|N^{x\succ y}| \geq |N^{y\succ x}|$, for every candidate $y \in M \setminus \{x\}$. We define symmetrically the (weak) Condorcet loser.

The plurality rule is considered to determine the winner of an election. Let $b_i \in M$ denote the ballot submitted by voter i and $b \in M^n$ denote the ballot profile, i.e., $b := (b_1,\dots,b_n)$. Let $b^\top$ denote the truthful ballot profile where all voters submit their sincere preferences, i.e., $b_i^\top = top(\succ_i)$ for every voter $i \in N$. The winner under plurality of the ballot profile b is $\mathcal{W}_P(b) \in \arg\max_{x\in M} s_x(b)$, where $s_x(b) := |\{i \in N : b_i = x\}|$ and a lexicographic tie-breaking, denoted by $\rhd$, is used if necessary. By abuse of notation, we sometimes write s_x instead of $s_x(b)$. Let I_n^m be the set of all possible candidates' scores under plurality, i.e., $I_n^m := \{s \in \mathbb{N}^m \mid \sum_{j=1}^m s_j = n\}$. By abuse of notation, we sometimes write $\mathcal{W}_P(s)$ to refer to the winner of a score vector s. Let $s^\top$ denote the candidates' scores in $b^\top$.

An election is given by the tuple $(N, M, \succ, \rhd)$. In this study, we examine the classical iterative voting model introduced by [22]. Initially, all voters vote truthfully, therefore the initial ballot profile b^0 is exactly the truthful ballot profile $b^\top$. Then they change their ballot strategically following a best response strategy which consists in supporting their preferred candidate within the set of so-called potential winners. A candidate y is a *potential winner* for voter i, at a given step where the current score vector is s, if i believes that voting for y will make candidate y the new winner, i.e., $s^{-i}_{\mathcal{W}_P(s^{-i})} - s^{-i}_y + \mathbb{1}_{\mathcal{W}_P(s^{-i}) \rhd y} \leqslant 1$, where s^{-i} denotes the score vector s without counting the current ballot b_i of voter i. Let PW_i^t denote the set of potential winners for voter i at step t, and PW^t the set of all potential winners at step t, i.e., $PW^t := \bigcup_{i \in N} PW_i^t$. When only a score vector is mentioned without a reference to a specific time step t, we may directly write $PW(s)$ to denote the set of potential winners according to a given score vector s.

We introduce the following notion to group the scores by the number of potential winners.

Definition 1. *Let S_n^j be the set of all score vectors in an n-voter election such that the union of potential winner sets over all voters contains exactly j candidates, i.e., $S_n^j = \{s \in I_n^m : |PW(s)| = j\}$.*

Note that $(S_n^j)_{j=1}^m$ forms a partition of I_n^m. Especially, S_n^1 corresponds to all score vectors with a unique potential winner. More precisely, for every score vector s in S_n^1, there exists a candidate which is the unique potential winner for all voters, and thus it is the winner in s.

We consider the following best response for each voter i at step t, where the current winner is denoted by w^{t-1}: i deviates from her current ballot b_i^{t-1} to another ballot b_i^t supporting candidate $y \in PW_i^{t-1} \setminus \{w^{t-1}\}$ if y is her most preferred candidate within PW_i^{t-1}. We then consider a best response dynamics which is defined via deviation sequences.

Definition 2 (Deviation sequence). *A sequence of strategy profiles $\langle b^0, b^1, \ldots, b^r \rangle$ is a deviation sequence for preference profile $\succ$ if:*

- *b^0 corresponds to the initial truthful ballot profile $b^\top$,*
- *for every step $t \in [r]$, state b^t results from a best response by exactly one voter from state b^{t-1}, i.e., for every step $t \in [r]$, there exists one voter $i \in N$ and one candidate $y \in PW_i^{t-1} \setminus \{w^{t-1}\}$ such that $y \succ_i z$ for every $z \in PW_i^{t-1}$, where $b_i^t = y$ and $b_j^t = b_j^{t-1}$ for every voter $j \in N \setminus \{i\}$,*
- *the sequence is maximal, i.e., b^r is an equilibrium where no voter has interest to change her ballot.*

We distinguish two types of strategic moves, one from a potential winner (FPW) (i.e., a deviation by voter i at step t from b_i^{t-1} to b_i^t where $b_i^{t-1} = x$ and $x \in PW^{t-1}$) and one from a non potential winner (FNPW) (i.e., a move by voter i at step t from b_i^{t-1} to b_i^t where $b_i^{t-1} = x$ and $x \notin PW^{t-1}$).

A deviation sequence is said to be *empty* if it is restricted to the initial ballot profile $\langle b^0 \rangle$ which is already an equilibrium.

From [22], we have an upper bound on the number of moves before convergence, in plurality iterative voting, which is given by $O(m \cdot n)$. We state below that this bound can be improved.

Proposition 1. *The number of moves in any deviation sequence is in* $O(m + n \cdot log(m))$.

Let us denote by $DS(\succ)$ the set of all possible deviation sequences for preference profile $\succ$. Indeed, since voters' deviations are performed sequentially, different deviation sequences can occur depending on which voter is selected to perform a strategic deviation at each step. The following example shows the potential diversity of iterative voting outcomes depending on the choice of the deviation sequence.

Example 1. Consider an election with five voters and four candidates, with voters' preferences as follows:

$$a \succ_1 c \succ_1 d \succ_1 b$$
$$b \succ_2 a \succ_2 c \succ_2 d$$
$$c \succ_3 b \succ_3 a \succ_3 d$$
$$d \succ_4 b \succ_4 a \succ_4 c$$
$$d \succ_5 c \succ_5 a \succ_5 b$$

When needed, a lexicographic tie-breaking rule is used. Initially, in the truthful preference profile, d is the winner. We show that each candidate can be the final winner in a different deviation sequence:

(a) If voter 2 deviates from b to a, then no other voter has an incentive to deviate afterwards and thus a is finally elected.
(b) If voter 3 deviates to b, followed by voter 5 who deviates to a and voter 4 who deviates to b, then no other voter has an incentive to deviate afterwards and thus b is finally elected.
(c) If voter 1 deviates to c, followed by voter 4 who deviates to b and voter 5 who deviates to c, then no other voter has an incentive to deviate afterwards and thus c is finally elected.
(d) If voter 3 deviates to b, followed by voter 1 who deviates to d, then no other voter has an incentive to deviate afterwards and thus d is finally elected.

Consequently, the notions of *possible* and *necessary* iterative winners naturally follow from the fact that different iterative winners can arise from different deviation sequences.

Definition 3 (Possible iterative winner). *A candidate x is a possible iterative winner for preference profile $\succ$ if there exists a deviation sequence $\langle b^0, b^1, \ldots, b^r \rangle \in DS(\succ)$ such that $w^r = x$.*

Definition 4 (Necessary iterative winner). *A candidate x is a necessary iterative winner for preference profile $\succ$ if, for every deviation sequence $\langle b^0, b^1, \ldots, b^r \rangle \in DS(\succ)$ we have $w^r = x$.*

By definition, a necessary iterative winner is also a possible iterative winner.

Let us provide below some observations to make the connections between these two concepts of iterative winner and the best response deviations based on potential winners. First of all, strategic moves are only possible towards potential winners. Thus, once a candidate leaves the set of potential winners, she can never return again.

Observation 1. *If a candidate x is a possible iterative winner for preference profile $\succ$, then there exists a deviation sequence $\langle b^0, b^1, \ldots, b^r \rangle \in DS(\succ)$ such that x is a potential winner all along the sequence: $\forall t \in \{0, 1, \ldots, r\}, x \in PW^t$. In particular, $x \in PW^0$.*

Moreover, from the definition of potential winner, if we remove one vote to a not currently winning potential winner, then she does not fulfill anymore the definition.

Observation 2. *Let us consider a deviation sequence $\langle b^0, b^1, \ldots, b^T \rangle \in DS(\succ)$ and the potential winner $x \in PW^t \setminus \{w^t\}$ such that the best response at step $t+1$ is a FPW move, i.e., the deviation from state b^t to reach b^{t+1} is performed by a voter $i \in N$ with $b_i^t = x$. Then $x \notin PW^{t+1}$.*

The concepts of possible and necessary iterative winners evaluate the outcomes of iterative voting processes from a qualitative perspective. Indeed, all deviation sequences must reach the same winner for the necessary iterative winner, whereas only one deviation sequence is required for the possible iterative winner. Another perspective is to take a more quantitative point of view. To this end, we will provide a probabilistic analysis of iterative voting outcomes.

Let Π^m be the set of all possible preference orders for m candidates. Let us denote as $C(n, \Pi^m)$ the probability distribution of drawing n preference orders from Π^m to constitute a preference profile $\succ \in (\Pi^m)^n$. Such a probability distribution is called a culture, and is simply denoted by C when the context is clear. The probability that a given event E occurs under culture C is denoted by $\mathbb{P}_C(E)$.

We will consider two commonly used cultures, namely *impartial culture (IC)* and *impartial anonymous culture (IAC)* [13].

Definition 5 (Impartial culture). *The impartial culture, called IC, draws every preference order $\succ_i$ independently from Π^m with uniform probability.*

Definition 6 (Impartial anonymous culture). *The impartial anonymous culture, called IAC, draws every preference profile $\succ$ from $(\Pi^m)^n$ with uniform probability.*

Let us now start our analysis of deviation sequences both from a qualitative and quantitative perspective.

4 Diversity of Iterative Winners

In this section, we will investigate how diverse iterative winners can be. We will first study the number of possible iterative winners and then focus on the extreme case with a necessary iterative winner, by analyzing the particular scenario where the deviation sequence is empty.

4.1 Number of possible iterative winners

We first observe that the iterative winner is determined when there are at most two potential winners.

Observation 3. *For any deviation sequence $\langle b^0, \ldots, b^r \rangle$, if $|PW^t| = 2$ for a given step $t \in \{0, 1, \ldots, r\}$, then the iterative winner of this sequence will be the winner of the pairwise comparison between the two candidates in PW^t.*

Observation 3 yields directly some straightforward corollaries:

Corollary 1. *If there exists a Condorcet winner c^*, and if $c^* \in PW^0$ with $|PW^0| = 2$, then c^* is the necessary iterative winner.*

Corollary 2. *A Condorcet loser can never be a possible iterative winner.*

Moreover, it can be used to bound the number of possible winners when there are only three candidates.

Proposition 2. *When $m = 3$, there exist at most two possible iterative winners.*

Proof. If there exists a Condorcet winner x then, since $m = 3$, there exists a weak Condorcet loser. In fact, x is winning every pairwise comparison therefore comparing the two other candidates tells us who is the Condorcet loser (resp., the two weak Condorcet losers). By Observation 3, the Condorcet loser (resp., the weak Condorcet loser, which is disadvantaged by the tie-breaking) cannot win. Hence, there can be at most two possible iterative winners.

If there does not exist a Condorcet winner then, since $m = 3$, we have to get either a strict or a weak Condorcet cycle of pairwise comparisons between these three candidates. In the case of a strict cycle, if we name y the initial winner, after the first strategic move we necessarily have a comparison between y and one of the other two candidates. However, with the strict Condorcet cycle one of these two need to lose against y thus, by Observation 3, this candidate cannot be elected. In the second case, the loser of the tie-breaking is also losing, helping us concluding the proof. □

Nevertheless, there exist situations where no candidate can be excluded from the set of possible iterative winners. We generalize below the observation made in Example 1 to show that for any number m of candidates, there exists a preference profile where all m candidates are possible iterative winners.

Proposition 3. *There exist elections where all m candidates are possible iterative winners, for every $m \geq 4$.*

Proof (Proof sketch). The case of $m = 4$ has already been shown in Example 1.

We provide here a general construction for every $m \geq 5$. Let us build a preference profile $\succ$ with $m+1$ voters and candidates $x_1, \ldots, x_m$, where the tie-breaking is given by $x_1 \rhd \cdots \rhd x_m$. To this purpose, we start with a preference profile $\succ^0$ where each voter $i \in [m-1]$ has the preferences $x_i \succ_i x_{i+1} \succ_i \cdots \succ_i x_m \succ_i x_1 \succ_i \cdots \succ_i x_{i-1}$, voter m has the preferences: $x_m \succ_m x_{m-1} \succ_m \cdots \succ_m x_2 \succ_m x_1$, and voter $m+1$ has the preferences $x_m \succ_{m+1} x_{m-2} \succ_{m+1} \cdots \succ_{m+1} x_{\lfloor \frac{m-1}{2} \rfloor + 1} \succ_{m+1} x_1 \succ_{m+1} x_2 \succ_{m+1} \cdots \succ_{m+1} x_{\lfloor \frac{m-1}{2} \rfloor} \succ_{m+1} x_{m-1}$. Then, we obtain our final profile $\succ$ from $\succ^0$ by swapping the positions of the adjacent candidates x_1 and x_m in agent 3 to agent $m-1$'s preference orders. For each candidate, one can exhibit a different deviation sequence which leads to her election.

4.2 Extreme case of necessary iterative winner

Let us now examine how frequently the initial ballot profile is already an equilibrium, leading hence to an empty deviation sequence, where the initial winner turns out to be the only possible iterative winner, and thus the necessary iterative winner. From [23], we know that, for each m, the proportion of truthful ballot profiles from which no voter has an incentive to deviate, tends to 1 as n increases. To better understand the behavior of iterative voting processes, even in small elections, we are particularly interested here in the rate of this convergence. While deriving an exact formula seems challenging, we propose, for each pair (m, n), an increasing lower bound in n for the proportion of equilibrium profiles. Let E_n^m be the set of all preference profiles $\succ$ that are equilibria. We start by providing some general results on the set of potential winners that will be used to establish the above-mentioned lower bound. Indeed, one way to deal with iterative voting is to track the set of potential winners over time t, i.e., PW^t.

The next lemma provides a characterization of potential winners:

Lemma 1. *Given a score vector $s \in I_n^m$, a candidate y is a potential winner for at least one voter $i \in [n]$, i.e., $y \in PW(s)$, if and only if all conditions (i) - (v) hold:*

(i) $\forall x \rhd y, s_x \leq s_y + 1$
(ii) $\forall x \rhd y, z \rhd y, s_x \leq s_y$ *or* $s_z \leq s_y$
(iii) $\forall x, z$ *such that* $y \rhd x, z, s_x \leq s_y + 1$ *or* $s_z \leq s_y + 1$
(iv) $\forall x$ *such that* $y \rhd x, s_x \leq s_y + 2$

(v) $\forall x, z$ *such that* $x \rhd y \rhd z, s_x > s_y \Rightarrow s_z \leq s_y + 1$

Lemma 1 allows to determine the size of S_n^m, as stated below.

Lemma 2. *The number of score vectors in* I_n^m *with* m *potential winners is equal to* m*, i.e.,* $|S_n^m| = m$.

Using the result of Lemma 2 as base case, we can finally determine the size of S_n^j for each $j \in [m]$.

Lemma 3. *For each* $k \in [m]$, $|S_n^{m-k}| = (m-k) \cdot \binom{n+k-2}{k}$.

We are now ready to present the main results of this section, which establish a lower bound on the probability that a preference profile (under impartial anonymous culture or impartial culture) is an equilibrium. We begin with the case of impartial anonymous culture.

Theorem 4. *Under impartial anonymous culture (IAC),* $\mathbb{P}_{IAC}(S_n^1)$ *increases with respect to* n *and* $\mathbb{P}_{IAC}(E_n^m) \geq \mathbb{P}_{IAC}(S_n^1)$.

Proof. As $S_n^1 \subset E_n^m$, we have $\mathbb{P}_{IAC}(E_n^m) \geqslant \mathbb{P}_{IAC}(S_n^1)$. Under IAC, we have $\mathbb{P}_{IAC}(S_n^1) = \frac{|S_n^1|}{|I_n^m|}$. By Lemma 3 (applied for $k = m-1$), we get $|S_n^1| = \binom{n+(m-1)-2}{m-1}$, and we have $|I_n^m| = \binom{n+m-1}{m-1}$. I we put them all together, we obtain after simplification:

$$\mathbb{P}_{IAC}(S_n^1) = \frac{n \cdot (n-1)}{(n+m-1)(n+m-2)}$$

It remains to be proven that $\mathbb{P}_{IAC}(S_n^1)$ increases with respect to n. Indeed, we have $\mathbb{P}_{IAC}(S_{n+1}^1) - \mathbb{P}_{IAC}(S_n^1) = \frac{2m-2}{(n+1)\cdot(n+2)\cdot(n+3)} > 0$ whenever $n > 0$ and $m > 2$.

We provide below a brief illustration of the growth rate of this lower bound.

Example 2. In an election with 3 candidates, the probability for a preference profile to be at equilibrium under IAC is at least 0.68 for 10 voters and at least 0.82 for 20 voters. In an election with 5 candidates, this probability is at least 0.49 for 10 voters and at least 0.69 for 20 voters.

We now establish an analogous result under impartial culture, starting with the following observation, based on the fact that there are n voters' preferences independently sampled from the same distribution, and we have m possibilities for the most preferred candidate of each voter.

Observation 5. *Whenever all voters' preferences are sampled with independent and identical random variables, then the resulting score vector* $s^\top$ *follows a multinomial law Multi*(q, n) *where* $q = (q_1, \ldots, q_m)$ *and* $q_j := \mathbb{P}_C(\{\mathcal{W}_P(s^\top) = j\})$, *for every* $j \in M$.

Under impartial culture, computing explicitly $\mathbb{P}_{IC}(S_n^1)$ becomes much more harder. Instead, we prove the existence of an increasing lower bound in n. The proof, based on a similar idea as the proof of Theorem 4 but required also new technical ideas, is deferred to the appendix.

Theorem 6. *Under impartial culture (IC),* $\mathbb{P}_{IAC}(E_n^m) \geq 1+\frac{m\cdot(m-1)}{2}\cdot(\phi(\frac{-2}{\sigma\cdot\sqrt{n}})-\phi(\frac{2}{\sigma\cdot\sqrt{n}}))$*, where* ϕ *is the cumulative distribution function of a standard Gaussian,* $\sigma = \sqrt{\frac{2}{m}}$ *and this probability is increasing with respect to* n*.*

This lower bound increases slowly compared to that of IAC:

Example 3. In an election with 3 candidates, 70 voters are needed for the probability to exceed 0.33, and 137 voters for it to exceed 0.5. In an election with 5 candidates, 1000 voters are needed for the probability to exceed 0.2.

We have examined two extreme cases in this section: one in which every candidate is a possible winner and another one in which only a single candidate can be a possible winner. In practice, we observe that a significant number of preference profiles have only one possible iterative winner, while the number of preference profiles with multiple possible iterative winners gradually decreases as the number of such winners increases (see the illustrations provided in the appendix).

5 Possible and Necessary Winner Problems

The situation analyzed in the previous section, where no deviation can occur from the initial ballot profile, is an extreme case of a scenario with a necessary iterative winner. In this section, we aim to go further on the recognition of configurations where given candidates are possible or necessary iterative winners, by investigating the complexity of the associated existence problems. More precisely, we will study the following decision problem PossibleIterativeWinner (resp., NecessaryIterativeWinner): *Given an election* $(N, M, \succ, \rhd)$ *and a candidate* $x \in M$*, is* x *a possible (resp., necessary) iterative winner?*

First of all, the two problems turn out to be equivalent when the initial potential winner set is limited to at most two candidates.

Proposition 4. *PossibleIterativeWinner and NecessaryIterativeWinner are equivalent and can be solved in polynomial time when* $|PW^0| \leq 2$*.*

Proof. If $|PW^0| = \{x\}$ then, by Observation 1, x is the unique possible–and thus necessary–winner.

If $PW^0 = \{x, y\}$ with $x = w^0$ then, by Observation 1, only x or y can be iterative winners. Since voters can only deviate to favor x or y and voters in $N^x \cup N^y$ have no incentive to deviate, candidate x (resp., y) is the unique possible–and thus necessary–iterative winner iff $|(N \setminus (N^x \cup N^y))^{x \succ y}| \geq |(N \setminus (N^x \cup N^y))^{y \succ x}|$ (resp., $|(N \setminus (N^x \cup N^y))^{y \succ x}| > |(N \setminus (N^x \cup N^y))^{x \succ y}|$).

Note that the equivalence between the two problems does not hold starting with three candidates in the initial potential winner set. Consider, e.g., the following preference profile with $n = 3$ voters and $m = 3$ candidates where $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 b \succ_3 a$, and a is the initial winner. If voter 2 (resp., voter 3) first deviates then c (resp., b) is the iterative winner. It follows that b and c are the two possible iterative winners, but none of them is a necessary iterative winner.

In addition of the non-equivalence of the two problems, even their complexity class differs. We first establish below that the necessary iterative winner problem can be solved in polynomial time.

Theorem 7. *NecessaryIterativeWinner is in P.*

Proof (Proof sketch). We will provide a polynomial number of conditions, which can be checked in polynomial time, on the preference profile $\succ$ to determine whether a given candidate y is a necessary winner. We distinguish the cases where y is the initial truthful winner w^0 or not. The case where $y = w^0$, a little more tedious, is deferred to the supplementary material.

Is candidate $y \neq w^0$ a necessary winner? Trivially, by Observation 1, if $y \notin PW^0$, then she is not a necessary iterative winner. Therefore, we assume from now on that $y \in PW^0$. We give some necessary conditions for y to be a potential winner along each possible deviation sequence:

(i) For all $z \in PW^0 \setminus \{w^0, y\}$ and all $i \in N^y$, we have $w^0 \succ_i z$: Otherwise, there exists a candidate $z \in PW^0 \setminus \{w^0, y\}$ and a voter $i \in N^y$ such that $z \succ_i w^0$. There exists then a deviation sequence where i is the first voter to deviate, from her initial ballot for y to a ballot for z. By Observation 2, y is not a potential winner anymore after this first step and thus, by Observation 1, y will not be the iterative winner in this deviation sequence.

(ii) Assume that (i) holds. For every candidate $z_1 \in M \setminus \{w^0, y\}$ and voter $i \in N^{z_1}$, we must have either $w^0 \succ_i z$ for every $z \in PW^0 \setminus \{w^0, z_1\}$, or $y \succ_i w^0$. Otherwise, there exist a candidate $z_1 \in M \setminus \{w^0, y\}$, a potential winner $z_2 \in PW^0 \setminus \{w^0, y, z_1\}$ and a voter $i \in N^{z_1}$ such that $z_2 \succ_i z$, for every $z \in PW^0 \setminus \{z_1, z_2\}$. There exists then a deviation sequence where i is the first voter to deviate, from her initial ballot for z_1 to a ballot for z_2 (that she prefers to w^0). Since w^0 was the initial winner, she is still a potential winner after this deviation. Therefore, there exists a second deviation in which a voter $j \in N^y$ deviates from her initial ballot for y to a ballot for w^0 (that she prefers over all potential winners other than y, by (i)). Thus, by Observations 1 and 2, y will not be the iterative winner in this deviation sequence.

Therefore, assume now that conditions (i) and (ii) hold.

– If, for every candidate $z_1 \in M \setminus \{w^0, y\}$ and voter $i \in N^{z_1}$, we have $w^0 \succ_i z$ for every $z \in PW^0 \setminus \{w^0, z_1\}$, then no deviation can occur. It follows that the initial winner w^0 will be the unique possible–and thus necessary–winner, implying that y cannot be a necessary winner.

– Otherwise, there exist a candidate $z_1 \in M \setminus \{w^0, y\}$ and a voter $i \in N^{z_1}$ such that $y \succ_i w^0$. In that case, by Observation 3, y is the unique possible–and thus necessary–iterative winner iff

$$|(\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{y \succ w^0}| > |(\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{w^0 \succ y}|$$

$\square$

In contrast, the possible iterative winner problem is NP-complete.

Theorem 8. *PossibleIterativeWinner is NP-complete.*

Proof (Proof sketch). The problem belongs to NP because, given a deviation sequence, we can check in polynomial time whether it is valid and eventually elects a target candidate t at equilibrium because the length of such a sequence is polynomially bounded (Proposition 1).

For hardness, we perform a reduction from a variant of Exact Cover by 3-Sets (X3C) [11] where, given a set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a set $S = \{S_1, S_2, \ldots, S_r\}$ of 3-element subsets of X, where each element x_i occurs in exactly three subsets of S (thus $r = 3q$), we ask whether there exists an exact cover, i.e., a subset $S' \subseteq S$ which is a partition of X.

For each element $x_i \in X$, we create a corresponding element-candidate y_i. For each subset $S_j \in S$, we create one candidate d_j and three subset-candidates s_j^1, s_j^2, and s_j^3 associated with the three elements of subset S_j. For each $\ell \in [2q]$, we create a candidate z_ℓ, supposed to correspond to the $2q$ elements of S which are not chosen for the partition of X. We additionally create five candidates, namely a, b, c, e, and t. The tie-breaking rule is given by the following linear order over the candidates: $a \rhd b \rhd c \rhd z_1 \rhd \cdots \rhd z_{2q} \rhd y_1 \rhd \cdots \rhd y_{3q} \rhd t \rhd d_1 \rhd \cdots \rhd d_{3q} \rhd e \rhd s_1^1 \rhd s_1^2 \rhd s_1^3 \rhd \cdots \rhd s_{3q}^1 \rhd s_{3q}^2 \rhd s_{3q}^3$.

For each element $x_i \in X$, we create $3q$ element-voters Y_i^ℓ, for $\ell \in [3q]$. For each $\ell \in [2q]$, we create $3q$ voters Z_ℓ^j, for $j \in [3q]$. Additionally, to allow all candidates to be potential winners, we create voters A^ℓ, B^ℓ, C^ℓ, D_j^ℓ, E^ℓ, $S_{j,k}^\ell$, and T^ℓ, for $j, \ell \in [3q]$ and $k \in [3]$. Finally, we create a candidate f and a voter F.

The preferences of these agents are described below, for each $i \in [3q]$ (where $s^\ell(x_i)$ stands for the subset-candidate s_j^k such that the k^{th} element of subset S_j is the ℓ^{th} occurrence of element x_i, when $\ell \in [3]$), for each $j, \ell \in [3q]$, $\ell' \in [2q]$, and $k \in [3]$:

Y_i^ℓ: $y_i \succ s^\ell(x_i) \succ a \succ t \succ [\dots]$	if $\ell \in [3]$
Y_i^ℓ: $y_i \succ a \succ t \succ [\dots]$	if $4 \leq \ell \leq 3q$
$Z_{\ell'}^j$: $z_\ell \succ c \succ y_1 \succ \dots \succ y_{3q} \succ s_j^1 \succ s_j^2 \succ s_j^3 \succ d_j \succ a \succ t \succ [\dots]$	
A^ℓ: $a \succ b \succ t \succ [\dots]$	
B^ℓ: $b \succ a \succ t \succ [\dots]$	
C^ℓ: $c \succ e \succ a \succ t \succ [\dots]$	
U^ℓ: $u \succ a \succ t \succ [\dots]$ for $(U, u) \in \bigcup_{j\in[3q]}\{(D_j, d_j), (S_{j,k}, s_j^k)\} \cup \{(E, e)\}$	
T^ℓ: $t \succ a \succ b \succ [\dots]$	
F: $f \succ z_1 \succ \dots \succ z_{2q} \succ y_1 \succ \dots \succ y_{3q} \succ t \succ a \succ b \succ [\dots]$	

By construction, in the truthful initial profile, there are exactly $3q$ votes for each candidate except f, and thus candidate a is winning.

We claim that there exists a subset $S' \subseteq S$ which is a partition of X iff there exists a deviation sequence which elects candidate t.

The global idea is that t must gain at least one vote by voter F. This is only possible if, previously, every candidate z_ℓ and y_i, for $\ell \in [2q]$ and $i \in [3q]$, loses at least one vote; and this via deviations done by $2q$ voters Z_ℓ^j and q voters Y_i^k, respectively, where voters Y_i^k deviate to subset-candidates that represent an exact cover of X.

Now that we have investigated whether an arbitrary candidate can be a possible or necessary iterative winner, it makes sense to focus on particularly desirable candidates.

6 About Electing the Condorcet Winner

Since electing the Condorcet winner is commonly considered as a desirable property for a voting rule, we now investigate the ability of the iterative voting process to elect it.

6.1 The Condorcet winner as an iterative winner

If a Condorcet winner exists, the natural question is whether she is guaranteed to be a possible or even a necessary iterative winner. We first study the question of a necessary iterative winner.

Proposition 5. *If $m = 3$ and the Condorcet winner is the initial winner, then she is also a necessary iterative winner.*

Proof. Let c^* be the Condorcet and initial winner. If no strategic move can be performed, we are done. Otherwise, the first strategic move of each deviation sequence cannot be neither towards nor from c^*, and by Observation 2, there are at most two potential winners after this move, c^* being one of them. Observation 3 implies that c^* is the winner of each sequence, hence the necessary winner.

However, the following example shows that if the Condorcet winner is not initially winning, she is not guaranteed to be the necessary iterative winner, even in the case where there are only 3 candidates.

Example 4. Let us consider the following profile:

$$b \succ_1 c \succ_1 a$$
$$a \succ_2 b \succ_2 c$$
$$c \succ_3 b \succ_3 a$$

Candidate b is the Condorcet but not initial winner (a initially wins by tie-breaking), and $PW^0 = M$. If voter 1 deviates from b to c, we get $PW^1 = \{a, c\}$. Since $b \notin PW^1$, she cannot win in this deviation sequence, therefore, she is not the necessary winner.

Similarly, the following example shows that if $m > 3$, then the Condorcet winner is not guaranteed to be the necessary iterative winner, and this is true even if she is the initial winner:

Example 5. Let us consider the following profile:

$$d \succ_1 c \succ_1 a \succ_1 b$$
$$a \succ_2 d \succ_2 c \succ_2 b$$
$$c \succ_3 b \succ_3 d \succ_3 a$$
$$b \succ_4 c \succ_4 a \succ_4 d$$
$$d \succ_5 a \succ_5 b \succ_5 c$$

Candidate d is the Condorcet and initial winner, and $PW^0 = M$. Let us exhibit a deviation sequence in which d is not winning. First, voter 4 deviates from b to c, making c the current winner and $PW^1 = \{a, c, d\}$. Then voter 5 deviates from d to a, yielding $PW^2 = \{a, c\}$. Since $d \notin PW^2$, she cannot win in this deviation sequence and is not the necessary winner.

On the other hand, the Condorcet winner is always guaranteed to be a possible iterative winner:

Proposition 6. *If the Condorcet winner is a potential winner of the truthful ballot b^0 (given a profile $\succ$), then she is a possible iterative winner.*

Proof. Let c^* be the Condorcet winner of given profile $\succ$, $c^* \in PW^0$. We show by construction that there exists a deviation sequence $\langle b^0, b^1, \ldots, b^r\rangle \in DS(\succ)$ such that $w^r = c^*$.

If $|PW^0| \leq 2$, then, by Corollary 1, c^* is a necessary and thus possible winner. Let us assume from now that $|PW^0| \geq 3$. In order to build a deviation sequence in which c^* is elected, we repeatedly use Observation 2 to rule out potential winners one by one, until we reach the situation where there are only two potential winners including c^* (hence, c^* is guaranteed to be elected). For each iteration t of the deviation sequence, there are two cases to distinguish:

- c^* **is not the current winner**: if there exists a potential winner $y \in PW^t$ and a voter i such that $b_i^t = y$ and $c^* \succ_i y$, then i can change her ballot from y to c^*, and by Observation 2, $y \notin PW^{t+1}$. Otherwise, all voters that vote for a potential winner at iteration t prefer all potential winners to c^*. By definition of the Condorcet winner, there are less than $\frac{n}{2}$ such voters, and there are less than $\frac{n}{2}$ voting for c^* (otherwise, we could not have more than 2 potential winners). Therefore, there exists a candidate $z \notin PW^t$ and a voter j such that $b_j^t = z$ and $c^* \succ_j y$ for each $y \in PW^t$. j can make a strategic move from z to c^*, making c^* the current winner. If after this move, $|PW^{t+1}| \leq 2$, let k be a voter such that $b_k^{t+1} = b_k^t = x \in PW^{t+1}$. As for any $y \in PW^{t+1} \setminus \{c^*, x\}$, we have $y \succ_k c^*$, the voter k can change her ballot from x to any other $y \in PW^{t+1} \setminus \{c^*, x\}$, so by Observation 2, $x \notin PW^{t+2}$.
- c^* **is the current winner**: if no strategic move is possible, we are done. Let us now assume the opposite. If there exists a voter i such that $b_i^t = x \in PW^t$, and $y \in PW_i^t$ such that $y \succ_i c^*$, then i can change her ballot for x to a ballot for y, and by Observation 2, $x \notin PW^{t+1}$. Otherwise, each voter casting her ballot for a potential winner at iteration t prefers c^* to any other potential winner. Then only FNPW moves are possible. Let j be a voter such that $b_j^t = z \notin PW^t$, and $y \in PW_j^t$ such that $y \succ_j c^*$. Then j can change her ballot for z to a ballot for y. If after this FNPW move, $PW^{t+1} = \{y, c^*\}$, c^* is a necessary (and thus possible) winner. Otherwise, there exists a candidate $x \in PW^{t+1}$, and we have assumed that each voter of x prefers c^* over all the other potential winners (different from x). In particular, there is a voter k such that $b_k^{t+1} = x$ who prefers c^* to the current winner y. k will then move to c^*, and by Observation 2, $x \notin PW^{t+2}$. □

6.2 Condorcet efficiency of the iterative rule

We have previously examined the conditions under which a Condorcet winner is a necessary or possible iterative winner. In this section, we go further by investigating how the iterative voting process affects the probability of electing the Condorcet winner.

More formally, we model iterative voting (under plurality) as a randomized voting rule, called *randomized iterative plurality*. Given the initial truthful score vector $s \in I_n^m$, we enumerate all possible deviation sequences and define the outcome as a probability distribution π^s over candidates, where for each $x \in M$, $\pi^s(x)$ denotes the proportion of sequences in which x is elected. Any branch has the same weight whatever its length. In particular, for a given score vector s, a candidate x is a possible iterative winner iff $\pi^s(x) > 0$, and a necessary iterative winner iff $\pi^s(x) = 1$.

For any given voting rule, the Condorcet efficiency (CE) is defined as the probability of electing the Condorcet winner when one exists:

Definition 7 (Condorcet efficiency). *When the Condorcet winner exists, we define the Condorcet efficiency as the probability to elect the Condorcet winner with respect to a voting rule.*

Note that for plurality, the Condorcet efficiency corresponds to $\mathbb{P}_C(c^* = \mathcal{W}_P(b^0) \mid c^* \text{ exists})$ while the Condorcet efficiency under randomized iterative plurality is to $\mathbb{P}_C(c^* = \mathcal{W}_P(b^r) \mid c^* \text{ exists})$, for any deviation sequence $\langle b^0, \dots, b^r \rangle$. To study whether the iterative voting increases the Condorcet efficiency it remains thus to study the sign of the value $\Delta CE = \mathbb{P}_C(c^* = \mathcal{W}_P(b^r) \mid c^* \text{ exists}) - \mathbb{P}_C(c^* = \mathcal{W}_P(b^0) \mid c^* \text{ exists})$.

This question has already been studied empirically by [15]. However, it has been done for a particular turn function which arbitrarily selects the voter allowed to deviate at each step. In contrast, our proof does not assume any turn function and considers all possible deviation sequences, via randomized iterative plurality.

In practice, we draw a preference profile under a certain culture C, and denote by C^*, similarly as [13], the culture associated with C that is reduced to preference profiles where the Condorcet winner exists.

Lemma 4. *Let C be a culture and c^* the Condorcet winner, when c^* exists, we have the following decomposition:* $\Delta CE = \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^r) \cap c^* \neq \mathcal{W}_P(b^0)) - \mathbb{P}_{C^*}(c^* \neq \mathcal{W}_P(b^r) \cap c^* = \mathcal{W}_P(b^0))$.

We start this formal work with a proof of the increase of the Condorcet efficiency under impartial anonymous culture (IAC).

Theorem 9.

Under IAC, the iterative voting process increases the Condorcet efficiency of plurality for any m, and n sufficiently larger than m.

Proof (Proof sketch). We denote $\mathbb{P}_{IAC}(\cdot \mid c^* \text{ exists})$ by $\mathbb{P}_{IAC^*}(\cdot)$ to simplify the notations, . Also, to shorten formulas and thus improve the readability of the proof, we use interchangeably the notations $\{|PW^0(s)| = k\}$ (resp., $|PW^0| = k$) and $s \in S^k$. To prove that $\Delta CE > 0$ whenever c^* exists, it suffices by Lemma 4 to show that

$$\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \cap c^* \neq \mathcal{W}_P(b^0))$$
$$> \mathbb{P}_{IAC^*}(c^* \neq \mathcal{W}_P(b^r) \cap c^* = \mathcal{W}_P(b^0))$$

The proof is organized as follows (technical details of each step are deferred to the appendix):

Upper bound on $\mathbb{P}_{IAC^*}(c^* \neq \mathcal{W}_P(b^r) \cap c^* = \mathcal{W}_P(b^0))$: We show that this term is upper bounded by:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC^*}(s \in S^k)$$

Lower bound on $\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \cap c^* \neq \mathcal{W}_P(b^0))$: We then show that this other term can be lower bounded by:

$$(\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC^*}(s \in S^2)$$

Intermediate step: Implication between IAC and IAC^*: To conclude, we now need to prove:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC^*}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC^*}(s \in S^2) \tag{1}$$

Since working directly under IAC^* is challenging, we rather prove the analogous inequality under IAC:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC}(s \in S^2) \tag{2}$$

We can actually prove that eq. (2) implies eq. (1).

Putting the bounds together under IAC: It remains to prove that eq. (1) holds when considering IAC, which can be done using Lemma 3 and a couple of combinatorial identities.

We now state the analogous result under impartial culture (IC).

Theorem 10. *Under IC, the iterative voting process increases the Condorcet efficiency of plurality for any m, and n sufficiently larger than m.*

Proof (Proof sketch). Following the same steps as in the proof of Theorem 9 but for IC, it remains to show:

$$\sum_{k=4}^{m} \mathbb{P}_{IC}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IC}(s \in S^2) \tag{3}$$

Since ϵ is going to 0 when n is large then we can just remove it. To prove eq. (3), we first prove the case of $m = 4$, and then we generalize its idea to $m > 4$.

If $\boldsymbol{m = 4}$, eq. (3) writes as:

$$\mathbb{P}_{IC}(s \in S^4) \leqslant \frac{1}{4} \cdot \mathbb{P}_{IC}(s \in S^2)$$

Let us denote by $S^{4\to 2}$ the set of score vectors with 2 potential winners obtained from some score vector of S^4 by transferring at most two votes between candidates. More formally,

$$S^{4\to 2} = \{s \in S^2 \mid \exists s' \in S^4 \text{ such that } s \text{ differs from } s' \text{ in 2 votes}\}$$

Also, for $s' \in S^4$, we denote by $S^{4\to 2}(s')$ all score vectors of $S^{4\to 2}$ built from s', i.e., $S^{4\to 2}(s') = \{s \in S^2 \mid s \text{ differs from } s' \text{ in 2 votes}\}$. To prove eq. (3) for $m = 4$, it is sufficient to prove that for each score vector $s \in S^4$, there exists a function $f^4 : S^4 \to [S^{4\to 2}]^8$ associating each score vector $s \in S^4$ with 8 different score vectors from $S^{4\to 2}(s)$ in a way that:

- $\forall s' \in f^4(s), \mathbb{P}_{IC}(s') \geqslant \frac{1}{2}\mathbb{P}_{IC}(s)$
- for each couple $s, s' \in S^4$, $f^4(s) \cap f^4(s') = \emptyset$.

The construction of f^4 and its generalization for $m > 4$ are deferred to the supplementary material.

7 Conclusion

In this article, we have examined the outcomes of iterative voting for the plurality rule under different aspects. We have particularly investigated the potential diversity of outcomes via the concepts of possible and necessary iterative winners. Although we may find instances where all candidates can be elected in some sequence of voters' deviations, we have experimentally shown that this configuration rarely occurs. Indeed, the most frequent situations are when a few different candidates turn out to be possible iterative winners. This is partly due to the existence of a necessary iterative winner, an event which is itself "biased" by the extreme scenario where no deviation is initially possible. We show that this extreme situation actually often occurs in our setting under impartial (anonymous) cultures.

In a computational point of view, the existence problem for a possible iterative winner is harder than for the necessary variant. It shows in a way that the kind of robustness created by the election of the same candidate at every sequence is easily detectable while more fluctuating configurations are difficult to predict. Beyond quantitative or computational results on possible outcomes, our analysis also helps provide theoretical insights on how beneficial manipulation can be. Indeed, we show that the frequency of election of the Condorcet winner is increased, when considering all possible iterative sequences with equal weights, under impartial (anonymous) cultures, compared to the single outcome of the initial plurality rule. This confirms and generalizes previous observations that were only made experimentally.

Our work opens several avenues for future work. While we have focused on a specific iterative voting setting, one could examine the impact of other types of strategic behaviors and voting rules [20]. Another natural direct extension would be to consider other–more realistic–voting cultures for probabilistic analyses, such as single-peaked ones, Mallows distributions, or even Polya-Eggenberger urns [5]. Finally, another more subtle study would be to analyze the strategic power of the voters (or their coalitions) on the iterative outcome, with respect to their position of deviation in the sequence or their preferences.

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Technical Appendix
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Proposition 1.

Proof. We identify the worst case scenario for the number of strategic moves in a deviation sequence. We start with a score vector in S_n^m. By Observation 2, the first move yields a score vector in S_n^{m-1}, in which the (unique) non-potential winner y_1 has less than $\frac{n}{m}$ votes. Since each voter i such that $b_i^1 = y_1$ can deviate to one of the $m-1$ remaining potential winners, we have at most $\frac{n}{m}$ FNPW deviations, each yielding, in the worst case, a new score vector in S_n^{m-1}. These are then followed by a FPW deviation that yields a score vector in S_n^{m-2}. We repeat the process–for each $k \in [m]$, when we reach a score of S_n^{m-k}, we have, in the worst case:

- $m-k$ potential winners, each obtaining approximately $\frac{n}{m-(k-1)}$ votes,
- one non-potential winner obtaining less than $\frac{n}{(m-(k-1))}$ votes,
- $k-1$ additional non-potential winners, each receiving zero votes.

We can thus perform at most FNPW moves and one FPW move before the next decrease of the number of potential winners. By **(author?)** [21], $|PW^t|$ can only decrease with t, so the process will terminate, and we will have at most

$$1 + \sum_{k=2}^{m} 1 + \frac{n}{m-k} = m + n \cdot \sum_{k=2}^{m} \frac{1}{m-k}$$

$$= m + n \cdot \sum_{l=0}^{m-2} \frac{1}{l} \leqslant m + n \cdot log(m)$$

strategic moves.

A Diversity of Iterative Winners

A.1 Number of possible iterative winners

Proposition 3.

Proof. The case of $m = 4$ has already been shown in Example 1. We will provide here a general construction for every $m \geq 5$.

We will build a preference profile $\succ$ with $m+1$ voters and candidates $x_1, \ldots, x_m$. To this purpose, we start with a preference profile $\succ^0$ where each voter $i \in [m-1]$ has the preferences $x_i \succ_i x_{i+1} \succ_i \cdots \succ_i x_m \succ_i x_1 \succ_i \cdots \succ_i x_{i-1}$, voter m has the preferences $x_m \succ_m x_{m-1} \succ_m \cdots \succ_m x_2 \succ_m x_1$, and voter $m+1$ has the preferences $x_m \succ_{m+1} x_{m-2} \succ_{m+1} \cdots \succ_{m+1} x_{\lfloor \frac{m-1}{2} \rfloor + 1} \succ_{m+1} x_1 \succ_{m+1} x_2 \succ_{m+1} \cdots \succ_{m+1} x_{\lfloor \frac{m-1}{2} \rfloor} \succ_{m+1} x_{m-1}$. Then, we obtain our final preference profile $\succ$ from $\succ^0$ by swapping the positions of the adjacent candidates x_1 and x_m in agent 3 to agent $m-1$'s preference orders.

For each candidate, we will describe a deviation sequence which leads to her election. When needed, we use the lexicographic tie-breaking rule.

- x_1: Voter $m-1$ deviates to x_1, then voter 2 deviates to x_m, and voter 3 deviates to x_1. Afterwards, the only potential winners are x_1 and x_m and, by construction, more voters prefer x_1 to x_m. It follows that the deviation sequence will finally elect x_1.
- x_i, for $2 \leq i \leq \frac{m-1}{2}$: Voter $i+1$ deviates to x_{i+2}, then voter $i-1$ deviates to x_i, and voter m deviates to x_{i+2}. Afterwards, the only potential winners are x_i and x_{i+2} and, by construction, more voters prefer x_i to x_{i+2}. It follows that the deviation sequence will finally elect x_i.
- x_i, for $\frac{m-1}{2} < i < m-1$: Voter $i-1$ deviates to x_i, then voter $m-1$ deviates to x_1, and then voter m deviates to x_i. Afterwards, the only potential winners are x_i and x_1 and, by construction, more voters prefer x_i to x_1. It follows that the deviation sequence will finally elect x_i.
- x_{m-1} if $m > 5$: Voter $m-2$ deviates to x_{m-1}, then voter 1 deviates to x_2, and then voter m deviates to $m-1$. Afterwards, the only potential winners are x_2 and x_{m-1} and, by construction, more voters prefer x_{m-1} to x_2. It follows that the deviation sequence will finally elect x_{m-1}.
- x_{m-1} if $m = 5$: Voter $m-2$ deviates to x_{m-1}, then voter $m+1$ deviates to x_1 (this is a best response because $x_{m-2} = x_{\lfloor \frac{m-1}{2} \rfloor + 1}$ when $m = 5$ and x_{m-2} is not a potential winner anymore because of the first deviation). Then, voter 2 deviates to x_{m-1}. Afterwards, the only potential winners are x_1 and x_{m-1} and, by construction, more voters prefer x_{m-1} to x_1. It follows that the deviation sequence will finally elect x_{m-1}.
- x_m: Voter 1 deviates to x_2, then voter 3 deviates to x_m. Afterwards, the only potential winners are x_2 and x_m and, by construction, more voters prefer x_m to x_2. It follows that the deviation sequence will finally elect x_m. □

A.2 Extreme case of necessary iterative winner

Lemma 1.

Proof. $\Leftarrow$ We suppose the conditions $(i) - (v)$ all hold, and we show that they are sufficient for y to be a potential winner for at least one voter. Conditions (i) and (iv) together imply that for each candidate x, we have $s_x \leq s_y + 2$:

- Suppose that there exists a candidate z such that $s_z = s_y + 2$. Condition (i) implies that $y \rhd z$. Therefore, y is a potential winner for each voter voting for z: indeed, we have $s_y + 1 = s_z - 1$, and y beats z by tie-breaking. Moreover:
  - condition (iii) ensures that for each x such that $y \rhd x$, $s_x \leq s_y + 1$, and in case of equality, y beats x by tie-breaking.
  - condition (v) implies that for each $x \rhd y$, we have $s_x \leq s_y < s_y + 1$, so y wins over x.

– Suppose now that for each candidate z, $s_z < s_y + 2$, and that there exists a candidate $x \rhd y$ such that $s_x = s_y + 1$. Therefore, y is a potential winner for each voter voting for x. Indeed, $s_x - 1 < s_y + 1$, so y wins over x if ever it receives one vote from x. Moreover:
  - condition (ii) implies that for each $x' \rhd y$, $x' \neq x$, we have $s_{x'} \leq s_y$, so $s_{x'} < s_y + 1$.
  - condition (v) implies that for each z such that $y \rhd z$, $s_z \leq s_y + 1$, and in case of equality, y beats z by tie-breaking.
– Finally, it is easy to see that whenever $s_y \geq s_x$ for all $x \rhd y$, and $s_z \leq s_y + 1$ for each z such that $y \rhd z$, y is a potential winner for all voters.

⇒ Now we need to prove that each of these conditions is actually necessary:

– if (i) does not hold, then there is a candidate x such that $s_x > s_y + 1$. Hence, even if one voter of x deviates to y, we will still have $s_x - 1 \geq s_y + 1$, and since x wins over y by tie-breaking, y can not be a potential winner for any voter.
– if (ii) does not hold, then there exist two candidates x and z with $x, z \rhd y$, such that $s_x > s_y$ and $s_y > s_y$. Therefore, $s_x \geq s_y + 1$ and $s_z \geq s_y + 1$, and as both x and z win over y by tie-breaking, y can not be a potential winner.
– if (iii) does not hold, there exist two different candidates x, z such that $y \rhd x, z$ and $s_x > s_y + 1$, $s_z > s_y + 1$. In other words, even if y obtains one more vote (possible from one of the candidates x and z), there will be at least one of y, z having a strictly higher score than y, and therefore y can not be a potential winner.
– if (iv) does not hold, there exists a candidate x such that $s_x > s_y + 2$, in other words, $s_x - 1 > s_y + 1$, so y can not be a potential winner.
– if (v) does not hold, there exist $x \rhd y$ and z such that $y \rhd z$ such that $s_x > s_y$ and $s_z > s_y + 1$. If y obtains one extra vote from z, we will still have $s_x \geq s_y + 1$, so x wins over y by tie-breaking. Otherwise, z wins over y. Hence, y can not be a potential winner. □

Lemma 2.

Proof. Let $n = qm + r, r \in \{0, \dots, m-1\}$, and $s \in S_n^m$. Let us denote by $\min_s$, resp. $\max_s$, the minimum, resp. maximum, score value in s. Without loss of generality, we can rename the candidates as $1, 2, \dots, m$ so that $i \rhd j$ iff $i < j$, and the score of candidate i corresponds to the i-th component s_i of s.

Since $s \in S_n^m$, the conditions $(i) - (v)$ of Lemma 1 must be satisfied for each component s_i of s. In particular, we can make the following three observations:

O_1 : $\min_s \geq q - 1$: let us assume for contradiction that $\min_s \leq q - 2$. Then the condition (iv) of Lemma 1 implies that $\max_s \leq q$ for each $i \in [m]$, so

$$\sum_{i=1}^{m} s_i \leq (q-2) + (m-1)q < n.$$

O_2 : $\max_s \leq q+2$: similarly to the previous case, let us assume for contradiction that $\max_s \geq q+3$. Then $\min_s \geq q+1$, and

$$\sum_{i=1}^{m} s_i \geq (q+3)+(m-1)(q+1) = qm+q+2 > n.$$

O_3 : It is easy to see that $\min_s \leq q$ and $\max_s \geq q$.

We are now ready to prove the statement by case distinction on r:

- $\mathbf{r=0}$: There are two possible values of $\min_s$:
  - $\min_s = q-1$. Then we necessarily have $\max_s = q+1$, otherwise, the sum of all components of s would be strictly less than n. Conditions (ii) and (iii) of Lemma 1 imply that there is a unique component of score $\max_s$, which implies that there is also a unique component of score $\min_s$ (to ensure that $\sum_{i=1}^{m} s_i = n$). The condition (v) implies that $s_1 = q-1$. We then need to choose the candidate $i \in \{2,\ldots,m\}$ such that $s_i = q+1$, all remaining candidates achieving the score of q–we note that for each possible value of i, the resulting vector satisfies Lemma 1. This yields $m-1$ vectors of S_n^m.
  - $\min_s = q$. We have then $\max_s = q$–otherwise, the sum of all components is greater than $mq = n$. There is a unique vector of this type, where all components are of value q.

  Put together, we have $|S_n^m| = (m-1)+1 = m$.
- $\mathbf{r \geq 1}$: The previous case implies that there is no $s \in S_n^m$ such that $\min_s = q-1$, and it is easy to see that $\max_s > q$. Hence, the above observations imply that $\min_s = q$, and $\max_s \in \{q+1, q+2\}$:
  - $\max_s = q+1$: there are r components of s of value $q+1$, and $(m-r)$ components of value q. The condition (ii) of Lemma 1 implies that for each $i \in [m]$ such that $s_i = q$, there is at most one $j < i$ such that $s_j = q+1$. Therefore, for each $i > (m-r)+1, s_i = q+1$–in other words, the $(r-1)$ last components of s equal $q+1$. There is one remaining component of value $q+1$ to be placed to one of the $(m-r)+1$ first positions. It is easy to check that regardless the choice, the score will satisfy all conditions of Lemma 1. Hence, there are $(m-r+1)$ scores of this type in S_n^m. Note that if $r=1$, we are done, and $|S_n^m| = m$.
  - $\max_s = q+2$ - note that this can only occur for $r \geq 2$. There are then $(r-2)$ components of value $q+1$, and $(m-r+1)$ components of value q. Note that there always exist (at least two) components of value q, so conditions (ii) and (iii) of Lemma 1 imply that there is a unique component of score $\max_s = q+2$. The condition (v) of Lemma 1 implies that for each pair i,j such that $s_i = q, s_j = q+1$, we have $i < j$. Similarly, the condition (i) implies that for each pair i,j such that $s_i = q, s_j = q+2$, we have $i < j$. In other words, the $(m-r+1)$ first components of s are all of value q, and we need to place the unique component of value $q+2$ to one of the remaining $(r-1)$ places. As previously, it is easy to check that each possible choice yields a score satisfying Lemma 1. Hence, there are $(r-1)$ scores of this type in S_n^m.

Putting both types together, we have $|S_n^m| = (m-r+1)+(r-1) = m$. $\square$

Lemma 3.

Proof. Let us start by defining the set of partial scores $\tilde{S}_n^{m-k}$ as follows: for each $s \in S_n^{m-k}$, we define $\tilde{s} \in \tilde{S}_n^{m-k}$ such that $\tilde{s}_i = s_i$ for each $i \notin PW(s)$, and for each $j \in PW(s)$, $\tilde{s}_j$ is a variable such that we have

$$\sum_{j \in PW(s)} \tilde{s_j} = n - \sum_{i \notin PW(s)} s_i.$$

Note that two (or more) scores $s, s' \in S_n^{m-k}$ can yield the same partial score of $\bar{S_n^{m-k}}$ - this happens if $PW(s) = PW(s')$ and each non-potential winner gets the same number of votes in both s and s'. We remove these duplicates from $\tilde{S}_n^{m-k}$. Lemma 2 implies that each partial score of $\tilde{S}_n^{m-k}$ can be completed into $(m-k)$ scores of S_n^{m-k}. Therefore, we have $|S_n^{m-k}| = (m-k)\cdot|\tilde{S}_n^{m-k}|$, and it remains to prove that $|\tilde{S}_n^{m-k}| = \binom{n+k-2}{k}$.

We proceed by induction on k. If $k=0$, Lemma 2 implies that $|S_n^{m-k}| = (m-k)$, and $|\tilde{S}_n^{m-k}| = 1 = \binom{n-2}{0}$. Let us now suppose that for given k, we have, for each $n \geqslant 0, \tilde{S}_n^{m-k} = \binom{n+k-2}{k}$, and let us prove that, for every $n \geqslant 0, \tilde{S}_n^{m-(k+1)} = \binom{n+k-1}{k+1}$.

We have:

$$\begin{aligned}\tilde{S}_{n+1}^{m-k} &= \binom{n+1+k-2}{k} = \frac{(n+k-1)!}{k!(n-1)!} = \\ &= \frac{(n+k-1)!(k+1)}{k!(k+1)(n-2)!(n-1)} = \\ &= \binom{n+(k+1)-2}{k+1}\cdot\frac{k+1}{n-1} = \\ &= \tilde{S}_n^{m-(k+1)}\cdot\frac{k+1}{n-1}\end{aligned}$$

Therefore, we get, for every $n \geqslant 0, \tilde{S}_n^{m-(k+1)} = \tilde{S}_{n+1}^{m-k}\cdot\frac{n-1}{k+1} = \binom{(n+1)+k-2}{k}\cdot\frac{n-1}{k+1} = \frac{(n+k-1)!(n-1)}{k!(k+1)(n-1)!} = \frac{(n+k-1)!(n-1)}{k!(k+1)(n-1)!} = \binom{n+k-1}{k+1}$, which ends the proof.

Theorem ??.

Proof. As in the proof of Theorem 4, we use $\mathbb{P}_{IC}(E_n^m) \geqslant \mathbb{P}_{IC}(S_n^1)$ We start by the following remark: "if for each pair of candidates $i, j \in M$, $|s_i - s_j| \geq 2$, then there is a unique potential winners in score s". Therefore, $\mathbb{P}_{IC}(S_n^1) \geq \mathbb{P}_{IC}(\forall i, j \in M, |s_i - s_j| \geq 2)$ There exist $\binom{m}{2} = \frac{m\cdot(m-1)}{2}$ pairs of candidates. We denote $X_k^{(i)}$ the random variable that equals if the k-th voter has voted for the i-th candidate, and 0 otherwise. We then denote $Y_k^{i,j} = X_k^{(i)} - X_k^{(j)}, \forall i \neq j$ the difference of those random variables such that $s_i - s_j = \sum_{k=0}^{n} Y_k^{i,j}$. Note that $(Y_k^{i,j})_{1\leqslant k\leqslant n}$ are independent, $\mathbb{P}_{IC}(Y_k = 1) = \mathbb{P}_{IC}(Y_k =$

$-1) = \frac{1}{m}$ and $\mathbb{P}_{IC}(Y_k = 0) = 1 - \frac{2}{m}$ Therefore, $\mathbb{P}_{IC}(\forall i, j \in M, |S_i - S_j| \geq 2) = \mathbb{P}_{IC}(\forall i, j \in M, |\sum_{k=0}^{n} Y_k^{i,j}| \geq 2)$. By Bonferroni's inequality, $\mathbb{P}_{IC}(\forall i, j \in M, |\sum_{k=0}^{n} Y_k^{i,j}| \geq 2) \geq \sum_{k=0}^{\frac{m\cdot(m-1)}{2}} \mathbb{P}_{IC}(|\sum_{k=0}^{n} Y_k^{i,j}| \geq 2) - (\frac{m\cdot(m-1)}{2} - 1)$. As all $Y_k^{i,j}$ follow the same law we have: $\sum_{k=0}^{\frac{m\cdot(m-1)}{2}} \mathbb{P}_{IC}(|\sum_{k=0}^{n} Y_k^{i,j}| \geq 2) - (\frac{m\cdot(m-1)}{2} - 1) = 1 + \frac{m\cdot(m-1)}{2} \cdot (\mathbb{P}_{IC}(|\sum_{k=0}^{n} Y_k^{i,j}| \geq 2) - 1)$. It remains to find a lower bound to $\mathbb{P}_{IC}(|\sum_{k=0}^{n} Y_k^{i,j}| \geq 2) = \mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} \geq 2) + \mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} \leqslant -2)$. Using Berry-Essen's theorem [3, 9] we get the following lower bounds: $\mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} \leqslant -2) \geq \phi(\frac{-2}{\sigma\cdot\sqrt{n}}) - \frac{C\cdot\rho}{\sigma^3\cdot\sqrt{n}}$ and $\mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} \geq 2) \geq \mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} > 2) = 1 - \mathbb{P}_{IC}(\sum_{k=0}^{n} Y_k^{i,j} \leqslant 2) \geq 1 - \phi(\frac{2}{\sigma\cdot\sqrt{n}}) - \frac{C\cdot\rho}{\sigma^3\cdot\sqrt{n}}$, where σ is the standard deviation, C is a constant, ρ the moment of order 3, ϕ is the repartition of a standard gaussian and $t \cdot \sigma\sqrt{n} = 2$, t from the original formula. However, if we compute ρ for $\sum_{k=0}^{n} Y_k^{i,j}$, we obtain $\rho = 0$, therefore simplifying the inequality as follows: $\mathbb{P}_{IC}(|\sum_{k=0}^{n} Y_k^{i,j}| \geq \phi(\frac{-2}{\sigma\cdot\sqrt{n}}) + 1 - \phi(\frac{2}{\sigma\cdot\sqrt{n}})$. As expected, this converges to 1 asymptotically in n since $\phi(0) = 0$. We finally get: $\mathbb{P}_{IC}(E_n^m) \geq 1 + \frac{m\cdot(m-1)}{2} \cdot (\phi(\frac{-2}{\sigma\cdot\sqrt{n}}) - \phi(\frac{2}{\sigma\cdot\sqrt{n}}))$. Note that $\phi(\frac{-2}{\sigma\cdot\sqrt{n}}) - \phi(\frac{2}{\sigma\cdot\sqrt{n}})$ can be verified to be negative since this the repartition of a standard gaussian. Therefore, this bound is increasing and goes to 1 asymptotically in n.

A.3 Illustrations

This subsection provides a detailed of the illustrations related to the number of potential winners.

To get quickly some first insights, we have drawn 1,000 elections, under impartial culture, where the preference profile is not an equilibrium, for each couple (m, n) with $m \in \{3, 4, 5\}$, and $5 \leq n \leq 15$, and we have computed the average number of possible iterative winners, represented in Figure 2. We note that, regardless the value of m, this average is rather low (less than 1.6 for all cases studied), and suggests a decreasing trend.

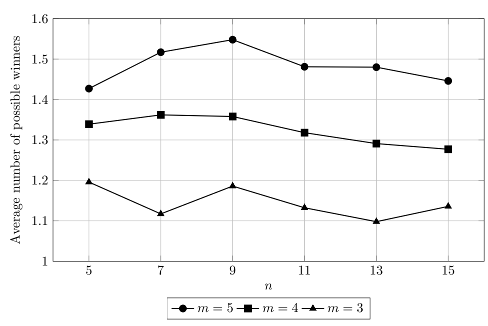

Fig. 1: Average number of possible iterative winners in function of n ($m = 3, 4, 5$)

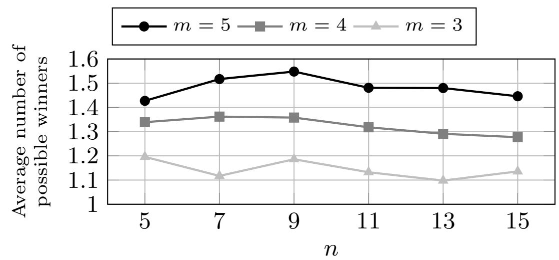

Fig. 2: Average number of possible iterative winners in function of n (for $m \in \{3,4,5\}$)

For a more in-depth view, we also provide in Figure 3 the distribution of the number of possible iterative winners of these randomly generated elections. We indeed observe that the vast majority of instances have a unique possible (and thus necessary) iterative winner. While for each m, there are still about 20% of instances with two possible iterative winners, the situations with more than two possible iterative winners, and in particular the extreme situation from Proposition 3, seem to be extremely rare.

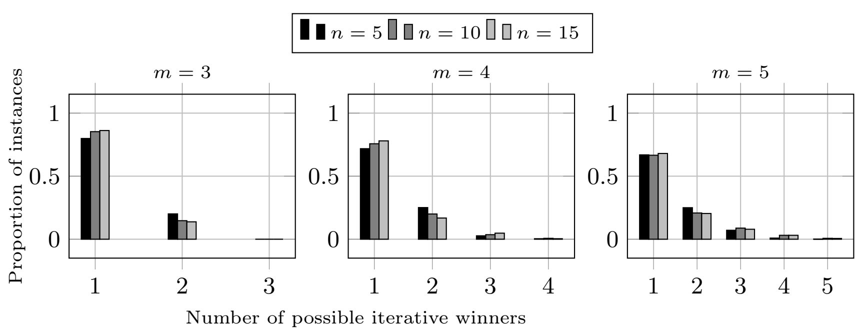

Fig. 3: Distribution of the number of possible iterative winners in function of n (for $m \in \{3,4,5\}$)

B Possible and Necessary Winner Problems

Theorem 7.

Proof. We will provide a polynomial number of conditions, which can be checked in polynomial time, on the preference profile $\succ$ to determine whether a given candidate y is a necessary winner. We distinguish the cases where y is the initial truthful winner w^0 or not.

Is candidate $y \neq w^0$ a necessary winner? Trivially, by Observation 1, if $y \notin PW^0$, then she is not a possible, and thus not a necessary iterative winner. Therefore, we assume from now on that $y \in PW^0$. Let us give some necessary conditions for y to be a potential winner along each possible deviation sequence:

(i) For all $z \in PW^0 \setminus \{w^0, y\}$ and all $i \in N^y$, we have $x \succ_i z$: Otherwise, there exists a candidate $z \in PW^0 \setminus \{w^0, y\}$ and a voter $i \in N^y$ such that $z \succ_i x$. There exists then a deviation sequence where i is the first voter to deviate, and she will do so from her initial ballot for y to a ballot for z. It follows from Observation 2 that y is not a potential winner anymore after this first step and thus, by Observation 1, y will not be the iterative winner in this deviation sequence, implying that y is not a necessary iterative winner.

(ii) Assume that (i) holds. For every candidate $z_1 \in M \setminus \{w^0, y\}$ and voter $i \in N^{z_1}$, we must have either $w^0 \succ_i z$ for every $z \in PW^0 \setminus \{w^0, z_1\}$, or $y \succ_i w^0$. Otherwise, there exist a candidate $z_1 \in M \setminus \{w^0, y\}$, a potential winner $z_2 \in PW^0 \setminus \{w^0, y, z_1\}$ and a voter $i \in N^{z_1}$ such that $z_2 \succ_i z$, for every $z \in PW^0 \setminus \{z_1, z_2\}$. There exists then a deviation sequence where i is the first voter to deviate, and she will do so from her initial ballot for z_1 to a ballot for z_2 (that she prefers to w^0). Since w^0 was the initial winner, she is still a potential winner after this deviation. Therefore, there exists a second deviation in which a voter $j \in N^y$ deviates from her initial ballot for y to a ballot for w^0 (that she prefers over all potential winners other than y, by (i)). Thus, by Observations 1 and 2, y will not be the iterative winner in this deviation sequence, implying that y is not a necessary iterative winner.

Assume that the conditions (i) and (ii) hold, and let us look closer to point (ii) where there are two cases to distinguish:

- If, for every candidate $z_1 \in M \setminus \{w^0, y\}$ and voter $i \in N^{z_1}$, we have $w^0 \succ_i z$ for every $z \in PW^0 \setminus \{w^0, z_1\}$, then no deviation can occur. It follows that the initial winner w^0 will be the unique possible–and thus necessary–iterative winner, implying that y cannot be a necessary iterative winner.
- Otherwise, there exist a candidate $z_1 \in M \setminus \{w^0, y\}$ and a voter $i \in N^{z_1}$ such that $y \succ_i w^0$. In that case, by Observation 3, y is the unique possible–and thus necessary–iterative winner iff

$$|(\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{y \succ w^0}| > |(\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{w^0 \succ y}|$$

.

Is candidate w^0 a necessary winner? If, for every candidate $z_1 \in M \setminus \{w^0\}$ and voter $i \in N^{z_1}$, we have $w^0 \succ_i z$ for every $z \in PW^0 \setminus \{w^0, z_1\}$, then no deviation can occur and w^0 is a necessary winner. Therefore, we assume from now on that there exists a candidate $z_1 \in M \setminus \{w^0\}$, a voter $i \in N^{z_1}$ and a candidate $z_2 \in PW^0 \setminus \{w^0, z_1\}$ such that $z_2 \succ_i w^0$, i.e., there is a voter with an incentive to deviate from the initial truthful profile b^0.

For a candidate $y \in PW^0 \setminus \{w^0\}$, let $PW^{1,y} \subseteq PW^0$ denote the set of potential winners in the strategy profile $b^{1,y}$ resulting from a best response electing candidate y, where a voter changes her initial ballot to a ballot for y, performed from the initial truthful profile b^0. Suppose that there exist a candidate $z_1 \in M \setminus \{w^0\}$, a potential winner $z_2 \in PW^0 \setminus \{w^0, z_1\}$, a voter $i \in N^{z_1}$ such that $z_2 \succ_i z$ for every $z \in PW^0 \setminus \{z_1, z_2\}$, a candidate $y \in PW^{1,z_2} \setminus \{w^0, z_2\}$ and a voter $j \in N^{w^0}$ such that $y \succ_i z$ for every $z \in PW^{1,z_2} \setminus \{w^0, y\}$. It means that there exists a deviation sequence where voter i is the first voter to deviate and she does so from her initial ballot for z_1 to a ballot for z_2 (that she prefers to w^0), and then voter j is the second voter to deviate and she does so from her initial ballot for w^0 to a ballot for y (that she prefers to the current winner z_2). It follows from Observations 1 and 2 that w^0 will not be the iterative winner in this deviation sequence, implying that w^0 is not a necessary iterative winner. Therefore, we assume from now on that, for every candidate $z_1 \in M \setminus \{w^0\}$, potential winner $z_2 \in PW^0 \setminus \{w^0, z_1\}$, voter $i \in N^{z_1}$ such that $z_2 \succ_i z$ for every $z \in PW^0 \setminus \{z_1, z_2\}$, we have all voters $j \in N^{w^0}$ who prefer z_2 over all potential winners in $PW^{1,z_2} \setminus \{w^0, z_2\}$.

Let Z denote the set of all potential winners to which there is a voter who has an incentive to deviate and $A(y)$ the set of voters having an incentive to deviate to $y \in Z$, i.e., $Z := \{y \in PW^0 \setminus \{w^0\} : \exists z_1 \in M \setminus \{w^0, y\},\ i \in N^{z_1} \text{ s.t. } y \succ_i z,\ \forall z \in PW^0 \setminus \{y, z_1\}\}$ and $A(y) := \{i \in N : \exists z_1 \in M \setminus \{w^0\} \text{ s.t. } i \in N^{z_1},\ y \succ_i z,\ \forall z \in PW^0 \setminus \{y, z_1\}\}$. By definition, we have $|A(y)| > 0$ for every $y \in Z$. If $|Z| = 1$ with $Z = \{y\}$, then the only first deviations that can occur are towards candidate y and no further deviation can then occur for a candidate different from w^0 or y and, by assumption, voters in N^{w^0} are satisfied by both candidates w^0 and y and thus do not deviate. It follows that w^0 is the unique possible–and thus necessary–iterative winner iff $(\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{w^0 \succ y} \geq (\bigcup_{z \in M \setminus \{w^0, y\}} N^z)^{y \succ w^0}$. Let us thus assume, from now on, that $|Z| > 1$.

By assumption, for every potential winner $z \in Z$, every voter $j \in N^{w^0}$ prefers z to any other potential winner $y \in Z \cap PW^{1,z}$. It follows that, for every candidates $z_1, z_2 \in Z$ such that $z_1 \neq z_2$, we have either $z_1 \notin PW^{1,z_2}$ or $z_2 \notin PW^{1,z_1}$. Note that both cannot hold simultaneously because for $z_2 \notin PW^{1,z_1}$ to hold, since $z_2 \in PW^0$, we need that $z_1 \rhd z_2$ or that $z_2 \rhd w^0 \rhd z_1$ while z_2 has one vote less than both z_1 and w^0 in the initial scores; under either condition z_1 is still a potential winner in the ballot profile b^{1,z_2} resulting from a best response from the truthful initial profile where z_2 gets one additional vote. Consequently, for every $z_1, z_2 \in Z$, we have either $z_1 \notin PW^{1,z_2}$ and $z_2 \in PW^{1,z_1}$ and all voters in N^{w^0} prefer z_1 to z_2, or $z_2 \notin PW^{1,z_1}$ and $z_1 \in PW^{1,z_2}$ and all voters in N^{w^0} prefer z_2 to z_1. We can thus assume, w.l.o.g., that $Z = \{z_1, \ldots, z_\ell\}$, with $z_t \notin PW^{1,z_{t'}}$, $z_{t'} \in PW^{1,z_t}$, and $z_t \succ_j z_{t'}$ for every voter $j \in N^{w^0}$ and every $1 < t < t' < \ell$.

For given indices $t_1 < t_2 < t_3 \in [\ell]$, let $A^{t_2,t_3}(t_1)$ denote the set of voters in $A(z_{t_1})$ who prefer z_{t_2} to w^0 and to z_t for all $t_3 \leq t \leq \ell$, i.e., $A^{t_2,t_3}(t_1) := \{i \in A(z_{t_1}) : z_{t_2} \succ_i w^0 \text{ and } z_{t_2} \succ_i z_t, \forall t_3 \leq t \leq \ell\}$. If there exist $t, t' \in [\ell]$ such that $t < t'$ with $|A(z_t) \cup \bigcup_{t'' \in [t'-1]} A^{t,t'}(t'')| > 1$, then there exists a deviation sequence where a voter $i_1 \in A(z_t)$ first deviates to a ballot for z_t, then a voter $j \in A(z_{t'})$ deviates to a ballot for $z_{t'}$, and another voter $i_2 \in A(z_t) \cup \bigcup_{t'' \in [t'-1]} A^{t,t'}(t'')$ then deviates to a ballot for z_t, creating a gap too important between the score of the current winner and the score of w^0, which thus cannot be a potential winner anymore. Consequently, by Observation 1, w^0 will not be the iterative winner in this deviation sequence, implying that w^0 is not a necessary winner.

Otherwise, it means that, for every candidate $z_t \in Z$, all voters in $A(z_t)$ prefer z_ℓ or w^0 to every $z_{t'} \in Z \setminus \{z_t, z_\ell\}$ (if not, $|A^{t',\ell}(t)| > 0$, and the previous condition would hold). Since, by definition, all voters in $N \setminus \bigcup_{t \in [\ell-1]} A(z_t)$ prefer w^0 to all candidates in Z, it follows that w^0 will be the unique possible–and thus necessary–iterative winner iff $(\bigcup_{z \in M \setminus \{w^0, z_\ell\}} N^z)^{w^0 \succ z_\ell} \geq (\bigcup_{z \in M \setminus \{w^0, z_\ell\}} N^z)^{z_\ell \succ w^0}$.

Theorem 8.

Proof. The problem belongs to NP because, given a sequence of voter strategic deviations, we can check in polynomial time whether it is valid and eventually elects a target candidate t at equilibrium because the length of such a sequence is polynomially bounded (see Proposition 1).

For hardness, we perform a reduction from Exact Cover by 3-Sets (X3C), a problem known to be NP-complete [11]. In an instance of X3C, we are given a set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a set $S = \{S_1, S_2, \ldots, S_r\}$ of 3-element subsets of X and we ask whether there exists an exact cover, i.e., a subset $S' \subseteq S$ of size $|S'| = q$ such that every element of X occurs in exactly one member of S', in other words, S' is a partition of X. We consider the variant of the problem, that is still hard, where each element x_i occurs in exactly three subsets of S, implying that $r = 3q$.

For each element $x_i \in X$, we create a corresponding element-candidate y_i. For each subset $S_j \in S$, we create one candidate d_j and three subset-candidates s_j^1, s_j^2, and s_j^3 associated with the three elements of subset S_j. For each $\ell \in [2q]$, we create an candidate z_ℓ, supposed to correspond to the $2q$ elements of S which are not chosen for the partition of X. We additionally create five candidates, namely a, b, c, e, and t. The tie-breaking rule is given by the following linear order over the candidates: $a \rhd b \rhd c \rhd z_1 \rhd \cdots \rhd z_{2q} \rhd y_1 \rhd \cdots \rhd y_{3q} \rhd t \rhd d_1 \rhd \cdots \rhd d_{3q} \rhd e \rhd s_1^1 \rhd s_1^2 \rhd s_1^3 \rhd \cdots \rhd s_{3q}^1 \rhd s_{3q}^2 \rhd s_{3q}^3$.

For each element $x_i \in X$, we create $3q$ element-voters Y_i^ℓ, for $\ell \in [3q]$, whose preferences are as follows for each $i \in [3q]$, where $s^\ell(x_i)$ stands for the subset-candidate s_j^k such that the k^{th} element of subset S_j is the ℓ^{th} occurrence of element x_i, when $\ell \in [3]$:

$$\begin{array}{ll} Y_i^\ell\colon y_i \succ s^\ell(x_i) \succ a \succ t \succ [\ldots] & \text{if } \ell \in [3] \\ Y_i^\ell\colon y_i \succ a \succ t \succ [\ldots] & \text{if } 4 \leq \ell \leq 3q \end{array}$$

For each $\ell \in [2q]$, we create $3q$ voters Z_ℓ^j, for $j \in [3q]$, with the following preferences:

$$Z_\ell^j: \quad z_\ell \succ c \succ y_1 \succ \cdots \succ y_{3q} \succ s_j^1 \succ s_j^2 \succ s_j^3 \succ d_j \succ a \succ t \succ [\ldots]$$

To allow all candidates to be potential winners, we create the voters A^ℓ, B^ℓ, C^ℓ, D^ℓ_j, E^ℓ, $S^\ell_{j,k}$, and T^ℓ, for $j, \ell \in [3q]$ and $k \in [3]$, with the following preferences:

$$
\begin{aligned}
&A^\ell\colon a \succ b \succ t \succ [\dots]\\
&B^\ell\colon b \succ a \succ t \succ [\dots]\\
&C^\ell\colon c \succ e \succ a \succ t \succ [\dots]\\
&U^\ell\colon u \succ a \succ t \succ [\dots]\\
&\quad\text{for } (U,u) \in \bigcup_{j\in[3q]}\{(D_j, d_j), (S_{j,k}, s^k_j)\} \cup \{(E,e)\}\\
&T^\ell\colon t \succ a \succ b \succ [\dots]
\end{aligned}
$$

We finally create an candidate f and a voter F with the following preferences:

$$F: \quad f \succ z_1 \succ \dots \succ z_{2q} \succ y_1 \succ \dots \succ y_{3q} \succ t \succ a \succ b \succ [\dots]$$

By construction, in the truthful initial profile, there are exactly $3q$ votes for each candidate except f, and thus candidate a is winning, thank to the tie-breaking rule.

We claim that there exists a subset $S' \subseteq S$ which is a partition of X iff there exists a sequence of voter strategic deviations which leads to the victory of candidate t.

$\Longrightarrow$: Suppose first that there exists a subset $S' \subseteq S$ which is a partition of X, say $S' = \{S_{j'_1}, \dots, S_{j'_q}\}$ where $j'_1 < \dots < j'_q$. By definition, each element x_i is covered by exactly one element of S', say that x_i is covered by the element of S' which contains the k_i^{th} occurrence of element x_i, for $k_i \in [3]$. We will thus let voter $Y_i^{k_i}$ deviate to subset-candidate $s^{k_i}(x_i)$. We will schedule these deviations with respect to the tie-breaking order $\rhd$, i.e., voter $X_i^{k_i}$ deviates before voter $X_{i'}^{k_{i'}}$, with $s^k_j := s^{k_i}(x_i)$ and $s^{k'}_{j'} := s^{k_{i'}}(x_{i'})$, iff $j > j'$, or $j = j'$ and $k > k'$. It follows that each candidate y_i loses one vote, while each candidate $s^k_{j'_\ell}$ gains one vote, for each $\ell \in [q]$ and $k \in [3]$, by decreasing order of indices.

Then, voter C^1 deviates from her vote for candidate c to a vote for candidate e, and thus c loses one vote. It follows that none of the candidates $y_1 \dots, y_{3q}$ and c are potential winners anymore, nor are any of the subset-candidates associated with elements of $S \setminus S'$.

Let us consider the set of non-chosen elements of S, i.e., $S \backslash S' = \{S_{j_1}, \dots, S_{j_{2q}}\}$ where $j_1 < \dots < j_{2q}$. For $\ell = 2q$ to $\ell = 1$, we let voter $Z_\ell^{j_\ell}$ deviate from candidate z_ℓ to candidate d_{j_ℓ}. This is a best response because none of the candidates $y_1 \dots, y_{3q}$, c, and $s^1_{j_\ell}$, $s^2_{j_\ell}$ and $s^3_{j_\ell}$ are potential winners.

Afterwards, voter F deviates from her vote for candidate f to a vote for candidate t. This is a best response because none of the candidates $z_1, \dots, z_{2q}$ and $y_1, \dots, y_{3q}$ are potential winners. Now let voter A^1 deviate from her vote for candidate a to a vote for candidate b. It follows that candidate a is not a potential winner anymore. If we then let, e.g., voter D^1_1 deviate from her vote for candidate d_1 to a vote for candidate t, then b and t are the only remaining potential winners, with $3q + 1$ and $3q + 2$ votes, respectively, while the other candidates which are less (resp., more) favored than t (resp., except b) have at most $3q+1$ (resp., $3q-1$) votes. Since there are more voters preferring t to b than the reverse, among the voters who do not currently vote for any of them, it thus leads to a sequence of voter deviations eventually electing candidate t at the equilibrium.

$\Longleftarrow$ *:* Suppose now that there exists a sequence of voter strategic deviations which leads to the victory of candidate t. First observe that, since all candidates (except f) have initially the same score, any iterative winner must gain at least one vote and thus must have at least $3q+1$ votes. Therefore, candidate t must gain at least one vote. Since candidates a, b, c, $z_1, \dots, z_{2q}$, $y_1, \dots, y_{3q}$ are more favored by the tie-breaking order $\rhd$ than t, none of them can gain a new vote before t gets one, because otherwise t would not be a potential winner anymore. All voters prefer a to t, except voters T^ℓ, who already vote for t, and voter F. Moreover, the only possibility for a to not be a potential winner before t can gain one vote, would be that some voter A^ℓ deviates, and the only possible deviation would be towards b, a contradiction. Therefore, we need that voter F deviates to t, and this is the only possible first deviation to t.

However, voter F prefers all candidates $z_1, \dots, z_{2q}$ and $y_1, \dots, y_{3q}$, initially potential winners, to candidate t. Therefore, we need for F to deviate to t as a first deviation to t, that none of the candidates $z_1, \dots, z_{2q}$ and $y_1, \dots, y_{3q}$ are potential winners, while t is still a potential winner. The only possible way to achieve this situation, is that every candidate z_ℓ and y_i, for $\ell \in [2q]$ and $i \in [3q]$, loses at least one vote.

Therefore, we need that at least one voter Y_i^ℓ, for some $\ell \in [3q]$, deviates from her current vote for y_i, for each $i \in [3q]$. The only possible deviation which can still enable the future election of t is by a voter Y_i^k for $k \in [3]$ towards $s^k(x_i)$. Let us construct the subset $S' \subseteq S$ such that all elements of S' correspond to subset-candidates $s^k(x_i)$ to which some voter Y_i^k deviates to, so that y_i is not a potential winner anymore, for each $i \in [3q]$. By definition of $s^k(x_i)$, it follows that S' covers all elements of X.

We also need that at least one voter Z_ℓ^j, for some $j \in [3q]$, deviates from her current vote for z_ℓ, for each $\ell \in [2q]$. To enable the first deviation of F to t, such a voter Z_ℓ^j should not deviate to c or $y_1, \dots, y_{3q}$, and thus none of these candidates should be a potential winner. It follows that all previously described deviations of voters Y_i^ℓ should occur before those of Z_ℓ^j. Moreover, the only possibility for c not being a potential winner anymore is that it loses one vote, with a deviation by a voter C^ℓ, for some $\ell \in [3q]$. Such a voter must deviate to candidate e. Then, by the tie-breaking order, none of the subset-candidates not chosen for deviation by voters Y_i^ℓ can be a potential winner anymore. Voter Z_ℓ^j can thus deviate to a subset-candidate s_j^k for $k \in [3]$ which has previously been chosen for deviation by a voter Y_i^ℓ or, if none of them has been chosen, to candidate d_j if not already the winner. However, it is not possible for the future election of t that Z_ℓ^j deviates to a subset-candidate s_j^k or to a candidate d_j which has already gained votes because, otherwise, such candidates would get at least

$3q+2$ votes and t would not be a potential winner anymore. It follows that each such voter Z_ℓ^j deviates to a different candidate d_j, and that no subset-candidate s_j^k, associated with the same element $S_j \in S$, has been chosen for deviation by voters Y_i^ℓ. Since there are $2q$ different such voters Z_ℓ^j associated with different elements $S_j \in S$ which are not part of S', it means that $|S'| = q$ and thus it is an exact cover of X.

C About Electing the Condorcet Winner

C.1 Condorcet efficiency of the iterative rule

Lemma 4.

Proof.

$$\begin{aligned}
\Delta CE &= \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^r)) - \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^0)) \\
&= \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* = \mathcal{W}_P(b^0)) \\
&\quad + \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0)) \\
&\quad - \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^0) \ \cap \ c^* = \mathcal{W}_P(b^r)) \\
&\quad - \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^0) \ \cap \ c^* \neq \mathcal{W}_P(b^r)) \\
&= \mathbb{P}_{C^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0)) \\
&\quad - \mathbb{P}_{C^*}(c^* \neq \mathcal{W}_P(b^r) \ \cap \ c^* = \mathcal{W}_P(b^0))
\end{aligned}$$

Theorem 9.

Proof. To prove that $\Delta CE > 0$ whenever c^* exists, it suffices by Lemma 4 to show that

$$\begin{aligned}
&\mathbb{P}_{IAC}(c^* = \mathcal{W}_P(b^r) \cap c^* \neq \mathcal{W}_P(b^0) \mid c^* \text{ exists}) \\
&> \mathbb{P}_{IAC}(c^* \neq \mathcal{W}_P(b^r) \cap c^* = \mathcal{W}_P(b^0) \mid c^* \text{ exists})
\end{aligned}$$

To simplify the notations, we denote $\mathbb{P}_{IAC}(\cdot \mid c^* \text{ exists})$ by $\mathbb{P}_{IAC^*}(\cdot)$. Also, to shorten formulas and thus improve the readability of the proof, we use interchangeably the notations $\{|PW^0(s)| = k\}$ (resp. $|PW^0| = k$) and $s \in S^k$.

Upper bound on $\mathbb{P}_{IAC^*}(c^* \neq \mathcal{W}_P(b^r) \ \cap \ c^* = \mathcal{W}_P(b^0))$*:* We first note that $\{c^* \neq \mathcal{W}_P(b^r) \ \cap \ c^* = \mathcal{W}_P(b^0)\} \subset \{\cup_{k=4}^m S_n^k\}$. Indeed, if $s(b^0) \in S_n^i$ for $i \leq 2$, then by Corollary 1, $c^* = \mathcal{W}_P(b^r)$. By Proposition 5, we also have $c^* = \mathcal{W}_P(b^r)$ when $s(b^0) \in S_n^3$ and $c^* = \mathcal{W}_P(b^0)$. Therefore,

$$\begin{aligned}
&\mathbb{P}_{IAC^*}(c^* \neq \mathcal{W}_P(b^r) \ \cap \ c^* = \mathcal{W}_P(b^0)) \\
&\leqslant \mathbb{P}_{IAC^*}(\cup_{k=4}^m \{|PW^0| = k\}) = \sum_{k=4}^m \mathbb{P}_{IAC^*}(|PW^0| = k)
\end{aligned}$$

The last equality is obtained because $\{|PW^0| = k\}_{4 \leqslant k \leqslant m}$ is a partition.

Lower bound on $\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0))$*:*

We have:

$$\begin{aligned}
&\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0)) \geqslant \\
&\geqslant \mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \ \cap \ s \in S^2) \\
&= \mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \mid s \in S^2) \\
&\cdot \mathbb{P}_{IAC^*}(s \in S^2) \\
&\geqslant \mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \ \cap \ c^* \in PW^0(s) \mid s \in S^2) \\
&\cdot \mathbb{P}_{IAC^*}(s \in S^2) \geqslant \\
&\geqslant \mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \mid c^* \in PW^0(s), s \in S^2) \\
&\cdot \mathbb{P}_{IAC^*}(c^* \in PW^0(s) \mid s \in S^2) \cdot \mathbb{P}_{IAC^*}(s \in S^2)
\end{aligned}$$

Let us now look closer to the two first terms of the last product:

(i) $\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \mid c^* \in PW^0(s), s \in S^2)$:
As the distribution over scores is uniform under IAC, if $PW^0 = \{c, c'\}$, then each of these two candidates has the same probability to be the initial winner. In other words, $\mathbb{P}_{IAC}(c = \mathcal{W}_p(b^0)) = \mathbb{P}_{IAC}(c' = \mathcal{W}_p(b^0)) = \frac{1}{2}$. Under IAC^*, the distribution over scores is biased in favor of the Condorcet winner c^* - we have

$$\mathbb{P}_{IAC^*}(c^* \neq W_p(b^0) \mid c^* \in PW^0(s), s \in S^2) = \frac{1}{2} - \epsilon,$$

with ϵ going to 0 when n grows and m is fixed. In addition, under assumptions that $s \in S^2$ and $c^* \in PW^0(s)$, by Corollary 1, c^* is the necessary winner, thus

$$\{c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0)\} = \{c^* \neq \mathcal{W}_P(b^0)\}$$

and hence

$$\begin{aligned}
&\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0) \mid c^* \in PW^0(s), s \in S^2) = \\
&= \frac{1}{2} - \epsilon.
\end{aligned}$$

(ii) $\mathbb{P}_{IAC^*}(c^* \in PW^0(s) \mid s \in S^2)$:
Again by the uniformity of scores under IAC, we have, for any m and any candidate c, for any m, $\mathbb{P}_{IAC}(c \in PW^0(s) \mid s \in S^2) = \frac{2}{m}$. Indeed, among the $\binom{m}{2}$ equally likely pairs of potential winners, c appears in $m-1$ of them. Under IAC^*, this distribution is again biased in favor of the Condorcet winner c^*, which yields

$$\mathbb{P}_{IAC^*}(c^* \in PW^0(s) \mid s \in S^2) \geqslant \frac{2}{m}$$

Put together, we get:

$$\begin{aligned}
&\mathbb{P}_{IAC^*}(c^* = \mathcal{W}_P(b^r) \ \cap \ c^* \neq \mathcal{W}_P(b^0)) \\
&\geqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC^*}(s \in S^2)
\end{aligned}$$

Intermediate step: Implication between IAC and IAC^*: To conclude the proof, we now need to prove that:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC^*}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC^*}(s \in S^2) \tag{4}$$

As working directly under the IAC^* distribution seems challenging, we will rather prove the analogous inequality under IAC:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC}(s \in S^2) \tag{5}$$

We can actually prove that Equation (5) implies Equation (4). Indeed, let us assume that Equation (5) holds. We note that

$$\mathbb{P}_{IAC}(c^* \text{ exists} \mid s \in \cup_{k \leqslant 4} S^k) \leqslant \mathbb{P}_{IAC}(c^* \text{ exists} \mid s \in S^2)$$

since the probability of Condorcet winner existence increases as the score becomes unbalanced. Therefore, we obtain

$$\sum_{k=4}^{m} \mathbb{P}_{IAC}(s \in S^k) \cdot \mathbb{P}_{IAC}(c^* \text{ exists} \mid \cup_{k \leqslant 4} S^k)$$

$$\leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC}(s \in S^2) \cdot \mathbb{P}_{IAC}(c^* \text{ exists} \mid s \in S^2)$$

Dividing by $\mathbb{P}_{IAC}(c^* \text{ exists})$, we get:

$$\frac{\sum_{k=4}^{m} \mathbb{P}_{IAC}(s \in S^k) \cdot \mathbb{P}_{IAC}(c^* \text{ exists} \mid \cup_{k \leqslant 4} S^k)}{\mathbb{P}_{IAC}(c^* \text{ exists})}$$

$$\leqslant \frac{(\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC}(s \in S^2) \cdot \mathbb{P}_{IAC}(c^* \text{ exists} \mid s \in S^2)}{\mathbb{P}_{IAC}(c^* \text{ exists})}$$

By the conditional Bayes's formula, we end up having:

$$\sum_{k=4}^{m} \mathbb{P}_{IAC}(s \in S^k \mid c^* \text{ exists})$$

$$\leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IAC}(s \in S^2 \mid c^* \text{ exists})$$

which is nothing but Equation (4):

Putting the bounds together under IAC: It remains to prove that Equation (5) holds. Using Lemma 3, we get:

$$\frac{\sum_{k=0}^{m-4}(m-k)\cdot\binom{n+k-2}{k}}{\binom{n+m-1}{m-1}}$$

$$\leqslant (\frac{1}{2}-\epsilon)\cdot\frac{2}{m}\frac{2\cdot\binom{n+m-4}{m-2}}{\binom{n+m-1}{m-1}}$$

With some algebraic simplifications and the use of the identity $k\cdot\binom{n+k-2}{k} = (n-1)\cdot\binom{n+k-2}{k-1}$ and perform a change of variable. This yields:

$$m\cdot\sum_{k=0}^{m-4}\binom{n+k-2}{k} - (n-1)\cdot\sum_{k=0}^{m-5}\binom{n+k-1}{k}$$

$$\leqslant (\frac{1}{2}-\epsilon)\cdot\frac{2}{m}\frac{2\cdot\binom{n+m-4}{m-2}}{m}$$

Using the following inequality (that can be easily proven by mathematical induction)

$$\sum_{k=0}^{M}\binom{A+k}{k} = \binom{A+M+1}{M}$$

with $A = n-2$ and $M = m-4$ for the first sum and $A = n-1$ and $M = m-5$ for the second, we get:

$$m\cdot\binom{n+m-5}{m-4} - (n-1)\cdot\binom{n+m-5}{m-5}$$

$$\leqslant (\frac{1}{2}-\epsilon)\cdot\frac{2}{m}\frac{2\cdot\binom{n+m-4}{m-2}}{m}$$

In other words:

$$m^2\cdot\binom{n+m-5}{m-4} - m\cdot(n-1)\cdot\binom{n+m-5}{m-5}$$

$$\leqslant (\frac{1}{2}-\epsilon)\cdot\frac{2}{m}2\cdot\binom{n+m-4}{m-2}$$

If we increase n, we will see that the inequality has to become true at some point. Indeed, ϵ becomes small as n grows and for m fixed and n large enough the left hand side becomes negative thanks to the second term while the second is increasing in n and is positive.

Theorem 10.

Proof. Following the same steps as in the proof of Theorem 9 but for IC, it remains to show:

$$\sum_{k=4}^{m} \mathbb{P}_{IC}(s \in S^k) \leqslant (\frac{1}{2} - \epsilon) \cdot \frac{2}{m} \cdot \mathbb{P}_{IC}(s \in S^2) \tag{6}$$

Since ϵ is going to 0 when n is large then we can just remove it.

To prove eq. (6), we first prove the case of $m = 4$, and then we generalizes its idea to $m > 4$.

Case of $m = 4$*:*

We need to prove that

$$\mathbb{P}_{IC}(s \in S^4) \leqslant \frac{1}{4} \cdot \mathbb{P}_{IC}(s \in S^2) \tag{7}$$

Let us denote by $S^{4\rightarrow 2}$ the set of scores with 2 potential winners obtained from some score of S^4 by transferring at most two votes between candidates. More formally,

$$S^{4\rightarrow 2} = \{s \in S^2 | \exists s' \in S^4 \text{ such that } s \text{ differs from } s' \text{ in 2 votes}\}.$$

Also, for $s' \in S^4$, we denote by $S^{4\rightarrow 2}(s')$ all scores de $S^{4\rightarrow 2}$ built from s', ie., $S^{4\rightarrow 2}(s') = \{s \in S^2 | s \text{ differs from } s' \text{ in 2 votes}\}$. To prove Equation (7), it is sufficient to prove that for each score $s \in S^4$, there exists a function $f^4 : S^4 \longrightarrow [S^{4\rightarrow 2}]^8$ association each score $s \in S^4$ with 8 different scores from $S^{4\rightarrow 2}(s)$ in a way that:

- $\forall s' \in f^4(s), \mathbb{P}_{IC}(s') \geqslant \frac{1}{2}\mathbb{P}_{IC}(s)$
- for each couple $s, s' \in S^4$, $f^4(s) \cap f^4(s') = \emptyset$.

We define below the function f^4. Let $s^4 \in S^4$. It remains to find 8 scores of $S^{4\rightarrow 2}(s)$ such that for each $s^2 \in S^{4\rightarrow 2}(s^4)$, $\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} \geqslant \frac{1}{2}$. We define f^4 so that all scores of f^4 are of the two following types:

- Type 1: $s^2 \in f^4(s^4)$ was built from s^4 by transferring two votes from a unique candidate j to two different candidates i and k.
- Type 2: $s^2 \in f^4(s^4)$ was built from s^4 by transferring one vote from candidates j, l to two remaining candidates i, k

We denote $s^4 = (s^4_1, s^4_2, s^4_3, s^4_4), s^4 \in S^4_n$, and $s^2 = (s^2_1, s^2_2, s^2_3, s^2_4), s^2 \in S^2_n$, and we have We have:

$$\mathbb{P}_{IC}(s^4) = \frac{n!}{s^4_1! \cdot s^4_2! \cdot s^4_3! \cdot s^4_4!}(\frac{1}{m})^n$$

and

$$\mathbb{P}_{IC}(s^2) = \frac{n!}{s^2_1! \cdot s^2_2! \cdot s^2_3! \cdot s^2_4!}(\frac{1}{m})^n.$$

Let us show that for each of these types, we have $\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} \geqslant \frac{1}{2}$ for n sufficiently large.

- Type 1: in all cases where we don't change the winner (resp. the winner changes), $|s_j^4 - s_i^4| \leq 1$ (resp. $|s_j^4 - s_i^4| \leq 2$) and $s_k^4 \geq q-1$ for each $k \in \{1,2,3,4\}$.
  Then we get:
  $$\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} = \frac{s_j^4(s_j^4-1)}{(s_i^4+1)(s_k^4+1)}.$$
  The smallest ratio is reached when $s_j^4 = q-1$, $s_i^4 = q$ and $s_k^4 = q$ if we don't change the winner and for $s_j^4 = q+2$, $s_i^4 = q$ and $s_k^4 = q+1$ otherwise.
  Therefore,
  $$\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} \geqslant \frac{(q-1)(q-2)}{(q+1)^2}$$
  We find this ratio is greater than $\frac{1}{2}$ for $q \geq 8$, ie., $n \geq 32$.
- Type 2: two votes are transferred from two different candidates j, l to two different candidates i, k. We get
  $$\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} = \frac{(s_j^4-1)(s_l^4-1)}{(s_i^4+1)(s_k^4+1)}.$$
  The same as in the previous cas, $s_p^4 \geq q-1$ for each $p \in \{1,2,3,4\}$, and for each $p, p' \in \{1,2,3,4\}$, we have $|s_p^4 - s_{p'}^4| \leq 2$. Therefore,
  $$\frac{\mathbb{P}_{IC}(s^2)}{\mathbb{P}_{IC}(s^4)} \geqslant \frac{(q-2)^2}{(q+2)^2},$$
  which is greater than $\frac{1}{2}$ for $q \geq 12$, ie., $n \geq 48$.

We will now build 8 scores of $f(s^4)$ as follows:

- We will create 5 scores of type 1, by distinguishing three sub-types:
  - 2 scores where the winner of s^4 gets one more vote and the looser of s^4 is not modified. We can then choose arbitrary which of the two remaining candidates x and y will get one more vote, and which one will loose two votes - indeed, each of these both choices yields a score of two potential winners, namely the winner of s^4 and the other candidate that gets one vote.
  - 1 scores where the the winner of s^4 gets one more vote, the third-ranked candidate of s^4 is not modified, and the second-ranked candidate looses two votes.
  - 2 scores vectors where the winner looses two votes. The candidate that is not modified needs to be the third or fourth ranked candidate of s^4 in order to ensure that the resulting score has two potential winners.
- Finally, we will create 3 more scores of type 2. The winner can not loose a vote because, depending of the number of votes of remaining candidates, we might reach a score with 3 potential winners. Therefore, the winner will get one more vote, and we have 3 choices for the second candidate to get one more vote, each of these yielding a score with 2 potential winners.

Moreover, all scores built by this construction are different, i.e $|f^4(s)| = 8$. Indeed, when starting from the same winner and adding one to her then the subtraction part differentiates the score of case 1 and 2.

The last thing to check is that $f^4(s) \cap f^4(s') = \emptyset$. Is is easy to see that for all cases where the winner gets one more vote (and in particular remains the winner), we can not have duplicates. Indeed, every score in S_n^4 yields a different winner, so winners will also be different in new scores. In the case we allow the winner to loose points (only for type 1) then this candidate who is now outside the potential winner set is last and characterized different new scores also.

General case: $m > 4$

We now explain how the construction of f^4 can be generalized for any $m > 5$. Let $m = 5$, for the case where the number of potential winners is 5, we can apply the same reasoning and we will have more cases to enumerate. For instance, for type 1, there is one more candidate that can loose 2 votes. Therefore, we can apply exactly the same idea of transformation as previously, namely $f^5 : S^5 \longrightarrow [S^{5 \to 2}]^h$, where $h > 8$ and $f^5(s) \cap f^4(s') = \emptyset$ because we start from different scores.

Let us show that $h - 8 \geq 2$ to preserve our probability ratio greater than $\frac{1}{2}$. Indeed, in case 2 we are able to build 3 more scores by taking a point to the fifth candidate, i.e. we now have $\binom{3}{2} = 3$ choices to subtract a point to two candidates. The case where this candidate has no vote can be treated separately. We see that these scores do not intersect. By recurrence, we apply the same reasoning when considering one more candidate and see that all the difficulty remains in the case of four potential winners for any m.