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Agreement Among Voting Rules Under Single-Peaked Preference Distributions

Vincent Mousseau[a], Henri Surugue[a,*] and Anaëlle Wilczynski[a]

[a]MICS, CentraleSupélec, Université Paris-Saclay

Abstract. Many different voting rules have been proposed in the literature and they can select very different alternatives. This naturally raises the question of whether this diversity in outcomes often occurs. Previous works have shown that the probability that voting rules agree on the same outcome is generally quite low under impartial culture. In this article, we use a similar probabilistic approach on single-peaked cultures, which are more structured and typically more realistic than impartial culture. We provide conditions for voting rules to agree under standard single-peaked cultures, and show that the probability of agreement between rather large families of voting rules is much higher under such cultures, with fast convergence of this probability with respect to the number of voters. We finally provide some insights on other structured preference distributions, observing that many exhibit similar convergence in agreement, including the Mallows' distribution. Our study reveals a tendency of several well-known voting cultures to bias the outcome of voting rules, which is worth knowing before conducting experiments on synthetic data.

1 Introduction

A major topic in voting theory is the design of good voting rules. However, the social choice literature is famous for impossibility theorems, e.g., Arrow's [1] or Gibbard-Satterthwaite [25, 41] theorems, basically stating that no perfect voting rule exists. Many different voting rules have been designed along the years, and a large body of literature is devoted to their axiomatic characterization [2]. In fact, different voting rules can select very different alternatives. However, does this behavior often occur? This question has been raised by many articles [24] which study the probability that different voting rules disagree on their outcome. Indeed, exploring the agreement among voting rules can help understand the similarity between voting rules, in an orthogonal perspective than the axiomatic study.

Most of the works on voting rules' agreement focus on the impartial (anonymous) culture, where each preference order (or score), is uniformly drawn from the whole set of linear orders over candidates. Such study is necessary because the impartial culture can arguably be seen as the most neutral. However, it does not capture real voters' preferences, which are usually far from being uniformly distributed. Moreover, most results on impartial culture highlight that voting rules rarely agree. Therefore, exploring more structured and realistic cultures may provide new insights on differences between voting rules. In this article, we will focus on cultures generating single-peaked preferences [5], which make sense in several contexts such as, e.g., political elections where a left-right axis can structure most voters' preferences. Even though single-peaked cultures are still far from being a perfect match to real data [18], they are much more realistic than impartial culture, so these models can be seen as a better approximation of the reality in some contexts.

In another point of view, studying agreement between voting rules under single-peaked cultures can also improve the understanding of such cultures. A key question in computational social choice, and in particular in voting theory, is how to generate relevant synthetic data for experiments on elections [8]. Conducting an empirical study via computer simulations can indeed be very useful to support or complement theoretical results for many voting problems, e.g., manipulation, winner determination, bribery and control, or the analysis of possible and necessary winners [9]. The ideal solution to perform experiments would be to use real-world data [14, 38, 39], see, e.g., the Preflib platform [33]. However, typically, we only have access to limited and context-dependent real-world data, which makes the experimental results potentially difficult to generalize. In contrast, using synthetic data allows to simulate elections of any size and to control the experiments' parameters. However, for experiments to be meaningful, we also need to simulate realistic elections, raising the question of a compromise between realism and flexibility. A large number of statistical cultures exist for generating elections [42]. Among them, single-peaked distributions are quite often used, as reported by Boehmer et al. [8]. Therefore, exploring voting rules' agreement under single-peaked cultures is relevant to better understand these commonly used cultures and better interpret experimental studies.

Let us illustrate possible issues in the interpretation of experiments. For instance, if one would like to compare how often different rules violate the majority criterion (i.e., a candidate ranked first by half of the voters should be elected), then experiments could be used. However, the conclusions may be very different depending on the voting culture used to generate synthetic data. In particular, using single-peaked cultures may lead to different conclusions compared to impartial culture, especially if the results on voting rules' agreement are very different. In particular, if two voting rules frequently agree under a given culture then the results will be similar because the voting rules are close under that culture, not because of the problem itself. In any case, knowing how the statistical tool works is a prerequisite for a good empirical study.

In this article, we study the probability of agreement of different voting rules under single-peaked cultures. Up to our best knowledge, this question has been surprisingly neglected for cultures more structured than impartial ones. One notable exception is the work of Chatterjee and Storcken [12] on unimodal profiles. We focus our

* Corresponding Author. Email: henri.surugue@centralesupelec.fr.

study on two well-known models to generate single-peaked elections: Walsh's [43] and Conitzer's [13] models. They consider different ways of uniformly drawing single-peaked preference orders: either uniformly within the whole single-peaked domain [43], or uniformly with respect to the peak candidate in the order [13].

We particularly examine positional scoring rules (PSRs), which compute scores for the candidates based on their position in the voters' preferences. This family covers many famous voting rules, such as k-approval rules like plurality or veto, and the Borda rule. We show that for both Walsh's and Conitzer's distributions, many PSRs tend to elect the median candidate(s) in the single-peaked axis, which turns out to be the asymptotic Condorcet winner, implying that these rules also agree with Condorcet-consistent rules. We also provide a lower bound on the speed of convergence to such a winner, meaning that this result holds for reasonable election sizes. We characterize these rules for both cultures and observe that this set is larger for Walsh's distribution, which is coherent with its definition. Conitzer's distribution seems to be more neutral toward the candidates, in the sense of probability to be elected. We further study this aspect by examining when single-peaked distributions are unbiased, i.e., when they do not favor any candidate with respect to a given voting rule.

Finally, we provide some insights on the agreement among voting rules under two other structured preference distributions: unimodal distributions, which include Mallows' cultures [32], where we complete Theorem 4.1 of Chatterjee and Storcken [12] to prove a rapid convergence to a large probability of agreement; and Pólya-Eggenberger urns [16], where we show that even if the probability of agreement remains high, the convergence toward one is not guaranteed.

2 Related Work

The question of agreement among voting rules was initiated by Gehrlein and Fishburn [21, 22] who give an explicit probability of agreement between two positional scoring rules in the case of three candidates under impartial culture. They prove that the probability of all scoring rules to agree in large elections is 0.5346. Many necessary conditions have then been derived to characterize the agreement of all positional scoring rules [36, 37, 40]. In particular, Merlin et al. [36] give the probability (i.e., 0.50116) under impartial culture that many rules (including positional scoring rules, elimination rules and Condorcet-consistent rules) agree on the same winner in the case of three candidates. This work was complemented via Monte-Carlo simulations by Lepelley et al. [30] for more than three candidates. Similar results with explicit formulas have been found under anonymous impartial culture for three candidates [19]. Most of these works focus on three candidates, sometimes four [28], under the impartial (sometimes anonymous) culture and try to provide explicit formulas. In contrast, we focus on single-peaked distributions with an arbitrary number of candidates and analyze the conditions of convergence toward the same outcome.

In another perspective, many works have studied the Condorcet efficiency of voting rules (see Gehrlein and Lepelley [24] for a survey), i.e., their probability to elect a Condorcet winner, which can be seen as exploring how much these rules agree with Condorcet-consistent rules. This question has also been investigated for structured cultures, such as impartial (anonymous) culture over the single-peaked domain [20, 29, 31], and Pólya-Eggenberger urns [23, 34] but, as far as we know, only for three candidates.

Another close question is the notion of consensus [17, 27], which is essentially setting a distance to find the closest election that satisfies consensus, i.e., the one where the minimum number of voters would disagree. Beyond voting rule agreement, the likelihood of the occurrence of voting paradoxes has been widely investigated [24, 45]. In addition, following the idea of asymptotic results, many studies have been conducted in machine learning, making the link between a voting rule and a maximum likelihood estimator [3, 11, 44]. In the same perspective, a work on the asymptotic probability of ties in elections was proposed [46]. While these directions may sometimes be outside of voting theory, it highlights the importance of our research question.

3 The Model

For any positive integer k, let $[k]$ denote the set $\{1, \dots, k\}$. Let N be a set of voters where $N = [n]$, and M be a set of m candidates where $M = \{x_1, \dots, x_m\}$. Each voter $i \in N$ has preferences over candidates represented by a linear order $\succ_i$ over M; the preference profile is denoted by $\succ = (\succ_i)_{i \in N}$. Let Π^m be the set of all possible preference orders for m candidates. For a given preference order $\succ_i \in \Pi^m$, the rank of candidate x in $\succ_i$ is denoted by $r_{\succ_i}(x)$, i.e., $r_{\succ_i}(x) := |\{y \in M : y \succeq_i x\}|$.

We consider a common preference restriction, namely *single-peakedness* [5]. A preference profile $\succ \in (\Pi^m)^n$ is single-peaked if there exists an axis $>$ on M such that, for every voter $i \in N$, and each triple of candidates $x > y > z$, we have $y \succ_i x$ or $y \succ_i z$. All along the article, we consider, w.l.o.g., an axis $>$ on M such that $x_1 > \dots > x_m$. Let $\Pi^m_>$ be the set of all possible single-peaked preference orders w.r.t. axis $>$ on M.

3.1 Voting Rules

A voting rule $\mathcal{F} : (\Pi^m)^n \to 2^M \setminus \{\emptyset\}$ selects a non-empty subset of candidates for each preference profile $\succ \in (\Pi^m)^n$. A scoring rule $\mathcal{F}$ is associated with a score function $s^{\mathcal{F}} : M \to \mathbb{R}$ and selects the candidates maximizing this score, i.e., $\mathcal{F}(\succ) \in \arg\max_{x \in M} s^{\mathcal{F}}(x)$ for every preference profile $\succ \in (\Pi^m)^n$.

A *positional scoring rule (PSR)* $\mathcal{F}$ is characterized by a positional score vector $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $\alpha_1 \geq \dots \geq \alpha_m$ and $\alpha_1 > \alpha_m$, in such a way that the winner of the election under $\mathcal{F}$ maximizes the sum of the position scores given by each voter according to the position of the candidate in the voter's preferences, i.e., $\mathcal{F}(\succ) \in \arg\max_{x \in M} \sum_{i \in N} \alpha_{r_{\succ_i}(x)}$ for every preference profile $\succ \in (\Pi^m)^n$. The *k-approval* voting rule, for $k \in [m-1]$, is a particular case of PSR where $\alpha_j = 1$ for all $j \in [k]$, and $\alpha_j = 0$ for all $k < j \leq m$. The *plurality* rule corresponds to the 1-approval rule and the *veto* rule is the $(m-1)$-approval rule. The *Borda* rule is the PSR characterized by an evenly spaced positional score vector, e.g., $\alpha = (m-1, m-2, \dots, 1, 0)$.

Instead of evaluating the candidates on their absolute position in the voters' preferences, other voting rules take into account pairwise comparisons of candidates. A candidate x is the *Condorcet winner* in preference profile $\succ \in (\Pi^m)^n$ if it beats all the other candidates in pairwise comparisons, i.e., $|\{i \in N : x \succ_i y\}| > |\{i \in N : y \succ_i x\}|$, for every candidate $y \in M \setminus \{x\}$. A *weak* Condorcet winner x is such that $|\{i \in N : x \succ_i y\}| \geq |\{i \in N : y \succ_i x\}|$, for every candidate $y \in M \setminus \{x\}$. In general, a (weak) Condorcet winner does not always exist. However, a weak Condorcet winner always exists when the preferences are single-peaked as well as a Condorcet winner when, additionally, m is odd [6]. A voting rule which always elects the Condorcet winner, when it exists, is called

Condorcet-consistent. Note that PSRs are not Condorcet-consistent [15].

3.2 Voting Cultures

Let us denote as $C(n, \Pi^m_{sub})$ the probability distribution of drawing n preference orders from $\Pi^m_{sub} \subseteq \Pi^m$ to constitute a preference profile $\succ \in (\Pi^m)^n$. Such a probability distribution $C(n, \Pi^m_{sub})$ is called a *culture*.

When voters' preferences are selected independently and identically distributed, the culture can be defined as drawing n preference orders $\succ_i$ from a given preference distribution $\pi^m : \Pi^m \to [0,1]$ with $\sum_{\succ_i \in \Pi^m} \pi^m(\succ_i) = 1$. The probability for a candidate x_j to be ranked at position $k \in [m]$ under preference distribution π^m is given by $\mathbb{P}^m_\pi(j,k) = \sum_{\succ_i \in \Pi^m : r_{\succ_i}(x_j)=k} \pi^m(\succ_i)$. Moreover, the probability for a candidate x to be ranked before a candidate y under preference distribution π^m is given by $\mathbb{P}^m_\pi(x \succ_i y) = \sum_{\succ_i \in \Pi^m : x \succ_i y} \pi^m(\succ_i)$. When the context is clear, the superscript m may be omitted.

Let $S^{\mathcal{F}}(x)$ denote the random variable giving the score of a candidate $x \in M$ for a voting rule $\mathcal{F}$. Let $\mathbb{E}_\pi[S^{\mathcal{F}}(x)]$ denote the expected score of candidate x for voting rule $\mathcal{F}$ under distribution π. For a PSR $\mathcal{F}$ characterized by a positional score vector α and a preference distribution π, the expected score of each candidate x is given by $\mathbb{E}_\pi[S^{\mathcal{F}}(x)] = \sum_{\succ_i \in \Pi^m} \pi(\succ_i) \cdot \alpha_{r_{\succ_i}(x)}$.

3.3 Convergence to the Expected Winners

When voters' preferences are identically and independently drawn w.r.t. distribution π and $\mathbb{E}_\pi[S^{\mathcal{F}}(x)]$ is finite for any $x \in M$, by the law of large numbers, the *expected winners* $\mathcal{W}_\pi(\mathcal{F})$ of $\mathcal{F}$ under π are $\mathcal{W}_\pi(\mathcal{F}) := \arg\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)]$. A candidate x is an *asymptotic (weak) Condorcet winner* under distribution π if $\mathbb{P}_\pi(x \succ_i y) > \frac{1}{2}$ (resp., $\mathbb{P}_\pi(x \succ_i y) \geq \frac{1}{2}$), for every $y \in M \setminus \{x\}$.

In addition to the guarantee of convergence to the election of expected winners, we provide below a lower bound on the probability that an expected winner actually wins, when we draw voters' preferences independently and identically with respect to a distribution π.

Theorem 1. *Consider a positional scoring rule $\mathcal{F}$ defined by a score vector α, and a preference distribution π over the set of candidates M. When the set of expected winners, defined as $\mathcal{W}_\pi(\mathcal{F}) = \arg\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)]$, is a singleton, i.e., $\mathcal{W}_\pi(\mathcal{F}) = \{x\}$, the probability that $\mathcal{F}$ elects x satisfies*

$$\mathbb{P}_\pi(x \in \mathcal{F}(\succ)) \geq L_\pi(\mathcal{F}),$$

where:

$$L_\pi(\mathcal{F}) := 1 - 2 \cdot \max_{y \in M \setminus \mathcal{W}_\pi(\mathcal{F})} \exp\left(\frac{-2n \cdot \left(\mu^{\mathcal{F}}_\pi(y) - \mathbb{E}_\pi[S^{\mathcal{F}}(y)]\right)^2}{\left(\max_j \alpha_j - \min_j \alpha_j\right)^2} \right)$$

$$\text{and } \mu^{\mathcal{F}}_\pi(y) := \frac{\max_{x \in M} \mathbb{E}_\pi[S^{\mathcal{F}}(x)] + \mathbb{E}_\pi[S^{\mathcal{F}}(y)]}{2}$$

We can thus deduce a lower bound for the speed of convergence for the agreement of several voting rules.

Corollary 2. *For two positional scoring rules $\mathcal{F}_1$ and $\mathcal{F}_2$ whose expected winner set under a preference distribution π is the same, i.e., $C := \mathcal{W}_\pi(\mathcal{F}_1) = \mathcal{W}_\pi(\mathcal{F}_2)$, the probability of their agreement for electing the same unique candidate from C is such that:* $\mathbb{P}_\pi(\mathcal{F}_1(\succ) = \mathcal{F}_2(\succ)) \geqslant \min\{L_\pi(\mathcal{F}_1), L_\pi(\mathcal{F}_2)\}$.

3.4 Single-Peaked Distributions

We particularly consider distributions based on the single-peaked domain. For a given axis $>$ over M, a culture $C(n, \Pi^m)$ is single-peaked if $C(n, \Pi^m) = C(n, \Pi^m_>)$.

Let us define the symmetry with respect to the single-peaked axis via the bijection $\tau : [m] \to [m]$ which associates with each candidate x_j its symmetric candidate $x_{\tau(j)}$ where $\tau(j) = m - j + 1$. A single-peaked preference distribution $\pi : \Pi^m_> \to [0,1]$ is said to be *symmetric* if $\mathbb{P}^m_\pi(j,1) = \mathbb{P}^m_\pi(\tau(j),1)$, for every candidate $x_j \in M$. Symmetric single-peaked distributions form a rather large family of single-peaked distributions which include, e.g., the distributions π such that $\mathbb{P}^m_\pi(x_j \succ_i x_{j+1}) = \mathbb{P}^m_\pi(x_{\tau(j)} \succ_i x_{\tau(j+1)})$ for every $j \in [\lfloor \frac{m}{2} \rfloor]$, but not only. Using symmetric single-peaked distributions turns out to be very natural, in order to derive experiments on the single-peaked domain, without any additional information than the single-peaked axis. In particular, two distributions are commonly used in the literature to sample single-peaked elections: Walsh's [43] and Conitzer's [13] distributions; they are symmetric and capture different types of impartial culture on the single-peaked domain. Roughly, the idea is either to uniformly draw every single-peaked preference order [43], or to uniformly draw every peak candidate and then construct the rest of the preference order by uniformly choosing the next candidate to rank between the closest available candidates on the single-peaked axis [13].

Definition 1 (Walsh's distribution)**.** *Walsh's distribution* $\pi_W : \Pi^m_> \to [0,1]$ *is such that* $\pi_W(\succ_i) = \frac{1}{2^{m-1}}$, *for every* $\succ_i \in \Pi^m_>$.

Definition 2 (Conitzer's distribution)**.** *Conitzer's distribution* $\pi_C : \Pi^m_> \to [0,1]$ *is such that* $\pi_C(\succ_i) = \frac{1}{m} \cdot \frac{1}{2^{\min\{r_{\succ_i}(x_1), r_{\succ_i}(x_m)\}-1}}$ *for every* $\succ_i \in \Pi^m_>$.

This definition adequately translates the algorithm proposed by Conitzer [13]. The peak is selected uniformly at random, corresponding to the $\frac{1}{m}$ term. Once the peak is fixed, the next candidate is chosen uniformly among the two candidates adjacent on the axis $>$, making the process dependent on the relative positions of the two extreme candidates. Specifically, once one of these two extreme candidates is selected, the rest of the ranking is completed by successively adding the remaining candidates on the same side with respect to the axis $>$.

In this article, we aim at understanding the behavior of voting rules under single-peaked distributions. In particular, we analyze the conditions under which PSRs agree, how the expected winners are located with respect to the single-peaked axis and whether they are asymptotic (weak) Condorcet winners.

4 The Single-Peaked Domain

Let us start with structural properties of the single-peaked domain. We first recall that $|\Pi^m_>| = 2^{m-1}$. We give below a useful observation on possible candidates' positions in single-peaked orders.

Observation 3. *Candidate x_j can never be ranked at a position $k > \max\{j, m-j+1\}$ in a single-peaked order.*

We continue our preliminary remarks on the structure of the single-peaked domain with the next lemma, already stated by Boehmer et al. [7], which will be useful to compute the probability for a candidate to be ranked at a given position.

Lemma 4 (Boehmer et al. [7]). *The number of single-peaked preference orders in $\Pi^m_>$ in which candidate x_j is ranked at position k is given by the following formula, for each $j, k \in [m]$:*

$$\mathscr{D}_m(j,k) = 2^{k-2}\left(\binom{m-k}{j-1} + \binom{m-k}{j-k}\right).$$ [1]

Let C^* denote the set of median candidates in the single-peaked axis, this set is a singleton in case m is odd and is a pair of candidates in case m is even, i.e.,

$$C^* := \begin{cases} \{x_{\lceil \frac{m}{2} \rceil}\} & \text{if } m \text{ is odd} \\ \{x_{\frac{m}{2}}, x_{\frac{m}{2}+1}\} & \text{if } m \text{ is even} \end{cases}.$$

These candidates play an important role in the single-peaked domain. We first show below that more preference orders rank them at good positions compared to the other candidates.

Lemma 5. *For every median candidate $x_c \in C^*$ and any other candidate $x_j \in M \setminus C^*$, there exists an index $\gamma_m(j) \in [\max\{j, m-j+1\}]$ such that $\mathscr{D}_m(c,k) \geq \mathscr{D}_m(j,k)$ for every $1 \leq k \leq \gamma_m(j)$ and $\mathscr{D}_m(j,k) > \mathscr{D}_m(c,k)$ for every $\gamma_m(j) < k \leq \max\{j, m-j+1\}$.*

Moreover, we show below that many natural single-peaked distributions favor the median candidates by tending to make them (weak) Condorcet winners.

Proposition 6. *Every symmetric single-peaked preference distribution makes the median candidate(s) asymptotic weak Condorcet winner(s). When m is odd, the unique median candidate is the asymptotic Condorcet winner under any symmetric single-peaked distribution π which assigns a positive probability to rank the median candidate first, i.e., $\mathbb{P}_\pi(c,1) > 0$ for $x_c \in C^*$.*

5 Walsh's Distribution

We first study Walsh's distribution (Definition 1), which corresponds to impartial culture on the single-peaked domain. The probability that a candidate appears at a given rank then directly follows from Lemma 4.

Observation 7. *The probability $\mathbb{P}_{\pi_W}(j,k)$ that candidate x_j is ranked at position k under Walsh's distribution, for each $j, k \in [m]$, is equal to $\mathbb{P}_{\pi_W}(j,k) = \frac{\mathscr{D}_m(j,k)}{2^{m-1}}$.*

We first establish that this distribution favors the median candidates since their expected score under every PSR is at least as large as the one of any other candidate.

Proposition 8. *For every PSR $\mathcal{F}$, the median candidates always belong to the expected winners of $\mathcal{F}$ under Walsh's distribution, i.e., $C^* \subseteq \mathcal{W}_{\pi_W}(\mathcal{F})$.*

Sketch of proof. One can show that the expected score of a median candidate is at least as large as the expected score of any other candidate, no matter the chosen positional score vector for the PSR. When comparing the expected score of a candidate $c \in C^*$ with the one of any other candidate $x_j \in M \setminus C^*$, we can restrict our attention, w.l.o.g., to the median candidate $x_c := x_{\lceil \frac{m}{2} \rceil} \in C^*$ and to any candidate x_j such that $j < \lceil \frac{m}{2} \rceil$ (by symmetry w.r.t. the single-peaked axis). By Observation 3, the expected score of a candidate x_j, for Walsh's distribution and a PSR $\mathcal{F}$ characterized by the positional score vector α, is given by $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] = \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$ and $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] = \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k$. One can then show to conclude that $\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \geq 0$. □

We now aim to characterize the PSRs for which the median candidates are the only expected winners. We identify them as the PSRs whose associated positional score vector α is such that there exists an index $\ell \in [\lfloor \frac{m}{2} \rfloor + 1]$ with $\alpha_\ell > \alpha_{\ell+1}$. We call them *first-prioritizing* PSRs. Note that all k-approval rules for $k \leq \lfloor \frac{m}{2} \rfloor + 1$ are first-prioritizing, as well as the Borda rule.

Theorem 9. *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Walsh's distribution, i.e., $\mathcal{W}_{\pi_W}(\mathcal{F}) = C^*$, iff $\mathcal{F}$ is first-prioritizing.*

Sketch of proof. Consider a PSR $\mathcal{F}$ characterized by a score vector α such that there exists an index $\ell \in [\lfloor \frac{m}{2} \rfloor + 1]$ for which $\alpha_\ell > \alpha_{\ell+1}$. We compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $c := \lceil \frac{m}{2} \rceil$ and $j < c$. By Observations 3 and 7 and Lemma 5, one can prove that:

$$\begin{aligned}
&\mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_W}[S^{\mathcal{F}}(x_j)] \\
&= \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \frac{\mathscr{D}_m(c,k)}{2^{m-1}} \cdot \alpha_k - \sum_{k=1}^{m-j+1} \frac{\mathscr{D}_m(j,k)}{2^{m-1}} \cdot \alpha_k \\
&= \frac{1}{2^{m-1}} \Big(\sum_{k=1}^{\gamma_m(j)} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k + \\
&\quad \sum_{k=\gamma_m(j)+1}^{m-j+1} (\mathscr{D}_m(c,k) - \mathscr{D}_m(j,k)) \cdot \alpha_k \Big) \\
&> \frac{\alpha_{\gamma_m(j)}}{2^{m-1}} \Big(\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor+1} \mathscr{D}_m(c,k) - \sum_{k=1}^{m-j+1} \mathscr{D}_m(j,k) \Big) = 0
\end{aligned}$$

Hence, the expected score of a median candidate is always greater than the one of any other candidate x_j.

For a PSR $\mathcal{F}$ characterized by a vector α where $\alpha_1 = \cdots = \alpha_\ell$, with $\ell > \lfloor \frac{m}{2} \rfloor + 1$, one can prove that both $x_{\lceil \frac{m}{2} \rceil} \in C^*$ and $x_{\lceil \frac{m}{2} \rceil - 1} \in M \setminus C^*$ are expected winners. □

By Proposition 6 and Theorem 9, the first-prioritizing PSRs tend to elect the (weak) Condorcet winner(s) under Walsh's distribution.

Corollary 10. *Under Walsh's distribution, all first-prioritizing PSRs and Condorcet-consistent rules asymptotically agree to elect the median candidates.*

We show a good lower bound for the convergence to the same outcome for a subset of first-prioritizing PSRs, which contains k-approval rules and the Borda rule.

Proposition 11. *For all k-approval voting rules that are first-prioritizing and the Borda rule, under Walsh's distribution, the probability of their agreement for electing one candidate from C^* is lower bounded by $L_\pi(\mathcal{F}_1)$ where $\mathcal{F}_1$ refers to the plurality rule.*

As an illustration, by Proposition 11, for $m = 5$, we have $\mathbb{P}_{\pi_W}(\mathcal{F}_1(\succ) = C^*) \geqslant 1 - 2e^{-\frac{n}{128}}$ and for $n = 600$, the probability of agreement is lower bounded by 0.98.

6 Conitzer's Distribution

We now analyze Conitzer's distribution (Definition 2), which considers a uniform distribution not on the whole single-peaked domain, as Walsh's distribution, but on the peak candidates of the single-peaked orders. It follows that the probability for a given candidate to be ranked at a given rank is a bit less direct, as already stated by Boehmer et al. [7].

[1] By convention, $\binom{n}{k} = 0$ when $k > n$ or $k < 0$.

Lemma 12 (Boehmer et al. [7]). *The probability that candidate x_j is ranked at position k under Conitzer's distribution, for each $j,k \in [m]$, is equal to $\mathbb{P}_{\pi_C}(j,k) = Q(j,k) + Q(m-j+1,k)$ where:*

$$Q(j,k) = \begin{cases} \frac{1}{2m} & \text{if } k < j \\ \frac{k}{2m} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

We first characterize the expected winners of all k-approval rules.

Proposition 13. *The expected winners of the k-approval rule $\mathcal{F}$ under Conitzer's distribution are:*

$$\mathcal{W}_{\pi_C}(\mathcal{F}) = \begin{cases} M \text{ if } k = 1 \\ \{x_k, x_{m-k+1}\} \text{ if } 1 < k \leq \lfloor \frac{m}{2} \rfloor + 1 \\ \{x_j \in M : \max\{j, m-j+1\} \leq k\} \text{ otherwise} \end{cases}.$$

Sketch of proof. We compute the expected score of a candidate x_j where, w.l.o.g., $j \in [\lceil \frac{m}{2} \rceil]$. By Lemma 12,

$$\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] = \begin{cases} \frac{2k}{2m} & \text{if } k < j \\ \frac{3k-1}{2m} + \frac{k}{2m} \cdot \mathbb{1}_{\{j = \lceil \frac{m}{2} \rceil\}} & \text{if } k = j \\ \frac{2j-1+k}{2m} & \text{if } j < k < m-j+1 \\ 1 & \text{if } k \geq m-j+1 \end{cases}$$

We can then derive the expected winners w.r.t. k. □

Hence, the only k-approval rule which tends to elect the median candidate(s) as unique expected winner(s) is $\lceil \frac{m}{2} \rceil$-approval (and $\frac{m}{2}+1$-approval if m is even).

We now characterize more precisely the PSRs which tend to elect the median candidate(s).

Theorem 14. *The median candidates are the unique expected winners of a PSR $\mathcal{F}$ under Conitzer's distribution iff the positional score vector α associated with $\mathcal{F}$ satisfies the following inequality, for every $1 \leq j < \lceil \frac{m}{2} \rceil$:*

$$\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \beta(m) + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} > \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell + \delta(j,m)$$

where $\beta(m) := (\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1}$ and $\delta(j,m) := (j-1)\alpha_j + (m-j+1)\alpha_{m-j+1}$.
A sufficient condition is $\beta(m) > \delta(j,m)$, for every $j < \lceil \frac{m}{2} \rceil$.

Sketch of proof. Consider a PSR $\mathcal{F}$ characterized by a score vector α. Let us compare a median candidate $x_c \in C^*$ and another candidate $x_j \in M \setminus C^*$ where, w.l.o.g., $j < c := \lceil \frac{m}{2} \rceil$. The median candidates are unique expected winners iff, for every $j < c$, we have $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_c)] - \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_j)] > 0$. One can prove that this is equivalent to $\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + (\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1} + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} > (j-1)\alpha_j + \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell + (m-j+1)\alpha_{m-j+1}$.

We always have $\sum_{\ell=j+1}^{\lceil \frac{m}{2} \rceil - 1} \alpha_\ell + \alpha_{\frac{m}{2}} \mathbb{1}_{\{m \text{ even}\}} \geq \sum_{\ell=\lceil \frac{m}{2} \rceil + 1}^{m-j} \alpha_\ell$. Hence, a sufficient condition is $(\lceil \frac{m}{2} \rceil - 1)\alpha_{\lceil \frac{m}{2} \rceil} + (\lfloor \frac{m}{2} \rfloor + 1)\alpha_{\lfloor \frac{m}{2} \rfloor + 1} > (j-1)\alpha_j + (m-j+1)\alpha_{m-j+1}$. □

We observe that the Borda rule satisfies the sufficient condition of Theorem 14, as well as $\lceil \frac{m}{2} \rceil$-approval (and $(\frac{m}{2}+1)$-approval if m is even), proving that these rules eventually elect the median candidates (as already observed in Proposition 13 for the approval rules). While Theorem 14 is not immediately interpretable, the following provides some intuition. Indeed, the underlying intuition is that the characterization corresponds to PSRs associated with a score vector $(\alpha_1, \cdots, \alpha_m)$ such that the first half of the scores is strictly greater than the second half but, for more than 4 candidates, not with too big a gap. More precisely, for m=3 and m=4, we must have $\alpha_2 > \alpha_3$ and $\alpha_2 > \alpha_3$ or $\alpha_3 > \alpha_4$, respectively, and for m=5, we must have $\alpha_2 > \alpha_3$ or $\alpha_3 > \alpha_4$ and $a_2 < 5 \cdot \alpha_3 - 4 \cdot \alpha_4$. Note that, in addition to Borda and $\lceil m/2 \rceil$-approval, this also includes, e.g., all PSRs such that $\alpha_i = 0$ if $i > \lfloor m/2 \rfloor + 1$ and $\alpha_2 < 2 \cdot \alpha_{\lfloor m/2 \rfloor + 1}$.

Corollary 15. *The median candidates are the unique expected winners of the Borda rule and the $\lceil \frac{m}{2} \rceil$-approval rule (as well as $(\frac{m}{2}+1)$-approval if m is even) under Conitzer's distribution.*

By Proposition 6 and Corollary 15, the Borda rule, $\lceil \frac{m}{2} \rceil$-approval, as well as all rules identified in Theorem 14 tend to elect the (weak) Condorcet winner(s).

Corollary 16. *Under Conitzer's distribution, the Borda rule, $\lceil \frac{m}{2} \rceil$-approval, and Condorcet-consistent rules asymptotically agree to elect the median candidates.*

As an illustration, when we apply Theorem 1 with Borda for $m = 5$, we have $\mathbb{P}_{\pi_C}(\mathcal{F}(\succ) = C^*) \geqslant 1 - 2e^{-\frac{9n}{3200}}$. For example, for $n = 2000$, we have a lower bound of 0.99 for the probability to elect the median candidate.

7 Unbiased Distributions

In this section, we aim to identify single-peaked distributions which do not favor any candidate by design, with respect to a given PSR. A preference distribution $\pi : \Pi^m \to [0,1]$ is said to be *unbiased* w.r.t. a given PSR $\mathcal{F}$ if all candidates are expected winners of $\mathcal{F}$ under π, i.e., $\mathbb{E}_\pi[S^{\mathcal{F}}(x)] = \mathbb{E}_\pi[S^{\mathcal{F}}(y)]$, for every $x, y \in M$. Note that the existence of an unbiased distribution w.r.t. a given PSR can be decided in polynomial time by solving a system of linear equations with real variables.

We first characterize the single-peaked distributions which are unbiased w.r.t. k-approval rules.

Theorem 17. *There exists an unbiased single-peaked distribution w.r.t. the k-approval rule iff k divides m.*

Sketch of proof. If k divides m, then there is an integer q such that $m = k \cdot q$. We partition the candidates M in q groups of size k where $X_j := \{x_{(j-1)k+1}, \ldots, x_{jk}\}$ for each $j \in [q]$, and $M = \bigsqcup_{j \in [q]} X_j$. For each group X_j, let P_j denote the non-empty set of single-peaked preference orders where the k candidates in X_j are ranked among the first k candidates, i.e., $P_j := \{\succ_i \in \Pi^m_{>} : r_{\succ_i}(x) \leq k, \forall x \in X_j\}$. One can prove that the distribution π such that $\sum_{\succ_i \in P_j} \pi(\succ_i) = \frac{1}{q}$ for each $j \in [q]$, and $\pi(\succ_i) = 0$ for all $\succ_i \in \Pi^m_{>} \setminus \bigcup_{j \in [q]} P_j$ is unbiased w.r.t. k-approval. □

From Theorem 17, no single-peaked distribution can be unbiased w.r.t. k-approval, for any $k > m/2$ when $m > 2$, which includes the veto rule (i.e., $(m-1)$-approval). Alternatively, there exists a family of single-peaked distributions which are unbiased w.r.t. the plurality rule (i.e., 1-approval), including Conitzer's distribution. In addition, we show that Conitzer's distribution is unbiased only w.r.t. plurality, leading to the following statement.

Proposition 18. *Conitzer's distribution is unbiased w.r.t. a positional scoring rule $\mathcal{F}$ iff $\mathcal{F}$ is the plurality rule.*

Sketch of proof. Suppose that Conitzer's distribution π_C is unbiased w.r.t. some PSR $\mathcal{F}$ defined by the positional score vector α. Since all candidates are expected winners of $\mathcal{F}$, we have in particular $\mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_1)] = \mathbb{E}_{\pi_C}[S^{\mathcal{F}}(x_2)]$, which leads to $\alpha_m = \frac{2}{m} \cdot \alpha_2 + \frac{m-2}{m} \cdot \alpha_{m-1}$. Because $\alpha_2 \geq \cdots \geq \alpha_m$, it implies that $\alpha_2 = \cdots = \alpha_m$, and $\alpha_1 > \alpha_m$, thus $\mathcal{F}$ corresponds to the plurality rule. □

In contrast, we prove that Walsh's distribution can never be unbiased because, no matter the chosen positional score vector, the expected score of a median candidate will always be strictly greater than the one of an extreme candidate in the single-peaked axis.

Proposition 19. *No PSR can make Walsh's distribution unbiased.*

We now consider a very degenerate distribution which only puts positive equal probability on the two extreme orders in the single-peaked domain.

Definition 3 (Polarized distribution)**.** *The polarized single-peaked distribution $\pi : \Pi^m_{\triangleright} \rightarrow [0,1]$ is defined as:*

$$\pi(\succ_i) = \begin{cases} \frac{1}{2} & \text{if } x_1 \succ_i \cdots \succ_i x_m \text{ or } x_m \succ_i \cdots \succ_i x_1 \\ 0 & \text{otherwise} \end{cases}.$$

Although it is degenerate, the polarized distribution is nevertheless symmetric and is the only single-peaked distribution which is unbiased w.r.t. the Borda rule.

Theorem 20. *A single-peaked distribution is unbiased w.r.t. the Borda rule iff it is the polarized distribution.*

Proof. The Borda rule is characterized by, e.g., the positional score vector $(m-1, m-2, \ldots, 1, 0)$. Under the polarized distribution, each candidate x_j can be ranked either at position j or at position $m-j+1$, with equal probability. It follows that the expected score of each candidate x_j is equal to $\frac{1}{2}(m-j) + \frac{1}{2}(j-1) = \frac{1}{2}(m-1)$. Therefore, the polarized distribution is unbiased w.r.t. the Borda rule.

Let us now prove that no other distribution is unbiased w.r.t. the Borda rule. Suppose that there exists a single-peaked distribution π which is unbiased w.r.t. the Borda rule. Observe that, globally, all the Borda scores that have been distributed to the candidates are equal to $\sum_{\succ_i \in \Pi^m_{\triangleright}} \pi(\succ_i) \cdot \sum_{x \in M}(m - r_{\succ_i}(x)) = \sum_{\succ_i \in \Pi^m_{\triangleright}} \pi(\succ_i) \cdot \frac{m(m-1)}{2} = \frac{m(m-1)}{2}$. Therefore, since all m candidates must have the same expected score, it must be equal to $\frac{m-1}{2}$. Let us denote by $\Pi^m_{\triangleright}(1)$ and $\Pi^m_{\triangleright}(m)$ the set of single-peaked orders where candidate x_1 and x_m are ranked last, respectively. We have $\Pi^m_{\triangleright} = \Pi^m_{\triangleright}(1) \sqcup \Pi^m_{\triangleright}(m)$. Candidates x_1 and x_m get zero points in $\Pi^m_{\triangleright}(1)$ and $\Pi^m_{\triangleright}(m)$, respectively. Since the maximum number of points to get is $(m-1)$, for x_1 and x_m to get an expected score of $\frac{m-1}{2}$, the distribution should be balanced between $\Pi^m_{\triangleright}(1)$ and $\Pi^m_{\triangleright}(m)$, i.e., we must have $\sum_{\succ_i \in \Pi^m_{\triangleright}(1)} \pi(\succ_i) = \sum_{\succ_i \in \Pi^m_{\triangleright}(m)} \pi(\succ_i) = \frac{1}{2}$. Moreover, for x_1 and x_m to reach an expected score of exactly $\frac{m-1}{2}$ on only half of the single-peaked orders, they must get $m-1$ points, i.e., be ranked at the first position, in the orders with positive probability in their half. Since both x_1 and x_m are ranked first in exactly one single-peaked order, i.e., in the extreme orders $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m$ and $x_m \succ_i \cdots \succ_i x_2 \succ_i x_1$, respectively, π must assign positive equal probability to exactly these two orders, leading to π being the polarized distribution. □

8 Other Structured Distributions

Finally, we explore structured preference distributions other than single-peaked ones in order to determine whether similar results can be reached. In particular, we study unimodal distributions, including the famous Mallows' distributions [32], introduced in voting theory by Goldsmith et al. [26], and Pólya-Eggenberger urn [16] introduced in voting theory by Berg [4].

8.1 Unimodal Distributions

The Kendall tau distance evaluates the similarity between two preference orders by counting the number of pairwise comparisons on which the two orders disagree, i.e., $dist_{KT}(\succ_i, \succ_j) = |\{(x,y) \in M^2 : x \succ_i y \text{ and } y \succ_j x\}|$, for every $\succ_i, \succ_j \in \Pi^m$. The frequency of a preference order $\succ_i \in \Pi^m$ in a preference profile $\succ \in (\Pi^m)^n$ is denoted by $f(\succ_i, \succ)$. A preference profile $\succ \in (\Pi^m)^n$ is *unimodal* [12] if there exists a mode $\succ^* \in \Pi^m$, i.e., a reference preference order, such that $f(\succ_i, \succ) > f(\succ_j, \succ)$ iff $dist_{KT}(\succ^*, \succ_i) < dist_{KT}(\succ^*, \succ_j)$, for every pair of preference orders $\succ_i, \succ_j \in \succ$. *Positively discriminating* rules [12] are social welfare functions which always return the mode as the outcome of the election. Both PSRs and Condorcet-consistent rules are positively discriminating.

We adapt the definition of unimodal profile to distributions. A preference distribution $\pi : \Pi^m \rightarrow [0,1]$ is said to be *unimodal* if there exists a mode $\succ^* \in \Pi^m$ such that $\pi(\succ_i) > \pi(\succ'_i)$ iff $dist_{KT}(\succ^*, \succ_i) < dist_{KT}(\succ^*, \succ'_i)$, for every pair of preference orders $\succ_i, \succ'_i \in \Pi^m$. We consider independent and identical voter preference drawings. By using the Glivenko-Cantelli theorem [10], we deduce that any unimodal distribution will asymptotically generate a unimodal profile, where PSRs and Condorcet-consistent rules agree to select the winner of the mode.

Corollary 21. *Under unimodal distributions, all PSRs and Condorcet-consistent rules asymptotically agree to elect the first-ranked candidate of the mode.*

We go further and give a bound for the speed of convergence toward agreement in terms of election size.

Proposition 22. *For a unimodal preference distribution π, the probability that all PSRs and Condorcet-consistent rules agree is lower bounded by $B_\pi := 1 - 2exp(-2n\varepsilon^2)$, for $\varepsilon := \min_{\succ_i, \succ_j \in \Pi^m} |\pi(\succ_i) - \pi(\succ_j)|$.*

A typical example of unimodal distributions are *Mallows' distributions* $\mathcal{M}^{\phi,\sigma}$, for given $\sigma \in \Pi^m$ and $\phi \in [0,1]$, defined by $\mathbb{P}_{\mathcal{M}^{\phi,\sigma}}(\succ_i) = \frac{1}{Z}\phi^{dist_{KT}(\succ_i,\sigma)}$ where $Z = \sum_{\succ_i \in \Pi^m} \phi^{dist_{KT}(\succ_i,\sigma)}$. Mallows' distributions are unimodal when $\phi < 1$. We give below an example of the speed of convergence under Mallows' distributions.

Example 1. *Under a Mallows' distribution $\pi^{\phi,\sigma}$, we get $\varepsilon = \phi^k \cdot (1-\phi)$ with $k := \max_{\succ} dist_{KT}(\sigma, \succ_i)$ and thus the bound for agreement is $B_{\pi^{\phi,\sigma}} = 1 - 2exp(-2n(\frac{\phi^k \cdot (1-\phi)}{Z})^2)$.*

If $\phi = 0.1$, $m = 3$ (then $k = 3$) and $n = 2{,}000{,}000$, we have $B_{\pi^{\phi,\sigma}} = 0.92$. If $\phi = 0.9$, $m = 3$ and $n = 400$, $B_{\pi^{\phi,\sigma}} = 0.97$. When more weight is given to orders close to the mode, voting rules agree faster than when the Mallows' distribution gets closer to impartial culture (i.e., $\phi = 1$).

8.2 Pólya-Eggenberger Urn

In the Pólya-Eggenberger urn model, we consider an urn initially containing $m!$ balls representing the $m!$ different preference orders from Π^m, i.e., each ℓ^{th} preference order from Π^m is initially drawn with probability $\beta_\ell = \frac{1}{m!}$. To draw our preference profile $\succ$ with n voters, for each voter, we draw a ball and assign to the voter the corresponding preference order and put it back into the urn with R additional balls with the same preference order and $R > 0$. We will assume $R = m! \cdot r$, for a given parameter r.

The following result generalizes the asymptotic result from Gehrlein [19] for three candidates under impartial anonymous culture (when $R = 1$).

Proposition 23. *Under the Pólya-Eggenberger urn culture, the probability that all PSRs asymptotically agree is lower bounded by* $\frac{1}{2}$ *if* $r < \frac{2}{3}$ *and* $m = 3$*, and by* $\frac{1}{4}$ *if* $r < \frac{1}{6}$ *and* $m = 4$.

We now analyze the agreement between plurality and Borda rule.

Proposition 24. *Under the Pólya-Eggenberger urn culture, the probability that plurality and Borda asymptotically agree is lower bounded by* $\frac{3}{4}$ *if* $r < \frac{2}{3}$ *and* $m = 3$*, and by* $\frac{3}{5}$ *if* $r < \frac{1}{6}$ *and* $m = 4$.

To give a comparison, under Walsh's distribution, for the agreement of plurality and the Borda rule to the election of median candidates C^*, we have a lower bound given by the plurality rule $\mathcal{F}_1$ (by Proposition 11) which is as follows: if $m = 4$, $\mathbb{P}_{\pi_W}(\mathcal{F}_1(\succ) = C^*) \geqslant 1 - 2e^{-\frac{n}{32}}$ is larger than $\frac{3}{5}$ when $n \geqslant 52$. Therefore, we are able to compare lower bounds and tell that the lower bound of Pólya-Eggenberger urn for $r < \frac{1}{6}$ is reached from $n \geq 52$ for the lower bound of Walsh's distribution.

We finally prove a positive probability of disagreement asymptotically for every pair of PSRs.

Proposition 25. *If the election is drawn with a Pólya-Eggenberger urn culture with* $R < 4$ *then every pair of positional scoring rules* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *asymptotically disagree with a positive probability, i.e.,* $\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_1(\succ) \neq \mathcal{F}_2(\succ)) > 0$.

This result means that any pair of positional scoring rules will disagree on a nonempty set asymptotically. Thus, we cannot achieve the same type of convergence results as in single-peaked distributions.

9 Conclusion

We have studied the probability of agreement of different voting rules under two single-peaked cultures, classically used for experiments in social choice, namely Walsh's and Conitzer's distributions. These distributions tend to favor the election of median candidate(s) in the single-peaked axis, and these candidates also turn out to be (weak) Condorcet winner(s), implying the agreement of several positional scoring rules (PSRs) with all Condorcet-consistent rules. This (weak) Condorcet efficiency holds in general for all symmetric single-peaked distributions, which are natural distributions for experiments when no additional information other than the single-peaked axis is available. We nevertheless observe that Conitzer's distribution is less biased toward the median candidates because it happens to be unbiased w.r.t. one PSR (namely plurality), contrary to Walsh's distribution. While these single-peaked distributions enable fast convergence to agreement, this is also the case for other structured distributions, such as unimodal ones, where the agreement is very general among voting rules and convergence is rapid. This behavior cannot be extended to Pólya-Eggenberger urns where the probability of disagreement is non-negligible, even if it remains high in some particular cases.

Our findings highlight that particular attention should be taken when using voting cultures for experiments in social choice. Indeed, since we identify cultures in which the agreement of different voting rules rapidly agree as the number of voters increases, conclusions drawn from experiments testing different voting rules for a problem should be interpreted with caution. One could imagine very different conclusions about a problem, not because of the problem itself, but because of the culture used: impartial cultures versus single-peaked cultures, for example. The take-home message of our results is to warn the community to be careful when using such cultures in experiments because some interpretations could be biased by the fact that voting rules mostly agree under these cultures.

Future work could consider bounds on the probability to agree in finite elections with Pólya-Eggenberger urn. The difficulty, however, lies in the dependent structure of this distribution. One idea could also be to consider nearly single-peaked distributions to bridge the gap between impartial and single-peaked cultures and be closer to real political elections. Furthermore, when voting rules asymptotically agree, we might conjecture that the probability of not satisfying certain axioms might also decrease as the election size increases. Finally, the same study could be done in a strategic model where voters can manipulate [35].

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